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High gain feedback, tolerance, and stabilization of linear systems

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The problem of stabilizing linear continuous-time systems is revisited with the objective of investigating the tolerance allowed in the implementation of stabilizing feedback controllers. It is shown that, for high gain feedback, this tolerance can be described by a cone in the feedback parameter space, called the "tolerance cone". The tolerance cone describes fractional (percentage) errors the feedback controller can tolerate without jeopardizing the internal stability of the controlled system. The larger the vertex angle of this cone, the more tolerance is available when implementing the controller. The vertex angle of the tolerance cone is determined by the proximity to singularity of a certain matrix derived from the controlled system.

1. Introduction

It is often assumed that feedback controllers must be designed and built with high accuracy. This is, of course, true in situations where the purpose of the feedback controller is to ensure performance accuracy of a closed loop control system, as is the case, for example, when designing a feedback amplifier with accurate gain. However, in cases where the main objective of the feedback controller is to stabilize a system, the situation is quite different; frequently, stability is maintained despite sizable inaccuracies in the feedback controller parameters. In this paper, we show that, often, there is a rather broad latitude in the selection of stabilizing feedback parameters, especially when high gain feedback is used. In extreme cases, this latitude is so wide that a stabilizing feedback controller can be obtained simply by assigning any large positive values to the feedback parameters, with no particular design plan. This observation seems consistent with practical design experience, where one frequently encounters systems that seem to work well with almost any choice of large feedback parameters.

The latitude available in assigning the parameter values of a stabilizing feedback controller depends on certain structural features of the system being controlled. For the case of linear static state feedback, we define in § 3 the notion of a *normalized controllability matrix*. The normalized controllability matrix is obtained by dividing the controllability matrix of the controlled system by another matrix derived from the same system. In addition, we introduce a measure that gauges how close a matrix is to being singular. This measure relates to the minimal angle between two columns of the matrix (needless to say, a square matrix becomes singular when the angle between two of its columns is zero). Finally, we use the notion of *tolerance* to describe the fractional (or percentage) accuracy required of an implementation, where lower tolerance means higher accuracy. In these terms, § 3 shows that the tolerance allowed in the implementation of a stabilizing high-gain state feedback controller diminishes as the normalized controllability matrix gets closer to being singular. In addition to providing a general quantitative assessment of the tolerance, this observation highlights an important role of the normalized controllability matrix: the closer this matrix is to being singular, the higher is the accuracy required of a stabilizing feedback controller.

The situation is very similar in the more general case of linear dynamic output feedback. From the given transfer matrix of the system being controlled, we derive a certain real matrix, called the *structure matrix*.

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The tolerance allowed in the implementation of a stabilizing high-gain dynamic output feedback controller is then determined by the proximity of this matrix to singularity.

In somewhat more technical terms, we study stabilization properties of high-gain feedback controllers using the following property of polynomials derived in § 2. Almost every monic polynomial with sufficiently large positive coefficients has all its roots in the open left half of the complex plane (here, a *monic polynomial* is a polynomial in which the highest power of the indeterminate appears with a coefficient of 1). We show that this set of polynomials gives rise to a set of stabilizing feedback controllers whose parameters form the unbounded end of an infinite cone in the feedback parameter space. In the case of linear static state feedback, the vertex angle of this cone is narrower the closer the normalized controllability matrix is to singularity. As a result, lower tolerance is available for the parameters of a feedback controller, when the normalized controllability matrix is close to being singular.

From a general perspective, the discussion of this paper can be viewed as a dual of Hammer (2004), where the implications of high forward gain were investigated. In that report, it was pointed out that, under rather general conditions, high forward gain allows high accuracy tracking with relatively little information about the system being controlled. In the present paper, we examine the implications of high gains in the feedback path. As it turns out, high gains in the feedback controller relieve some of the burden of accuracy required to stabilize a system. Indeed, for a "typical" system (i.e., a system whose normalized controllability matrix is not close to being singular), there is a very substantial leeway in assigning the parameters of a stabilizing feedback controller. In other words, the same system can be stabilized by a relatively large family of feedback controllers. This feature achieves its full power when high gain feedback controllers are employed.

The paper concentrates on linear continuous-time systems, but the basic principles can be extended to more general classes of systems. Section 2 examines a certain class of monic polynomials with large positive coefficients, showing that polynomials belonging to this class have all roots in the open left half of the complex plane. This feature is then used to investigate the tolerance requirements of linear static state feedback (§ 3) and linear dynamic output feedback (§ 4) under conditions of high feedback gains.

Investigations of linear control systems have taken a variety of different approaches over the last several decades, with each approach contributing an important aspect to the present understanding of the theory and practice of linear control. The current discussion employs mostly a selection of classical control techniques (e.g., Newton *et al.* (1957)) combined with algebraic techniques based on the theory of matrices over principal ideal domains (Rosenbrock (1970), Hammer (1983a–c)). These techniques are employed to examine the consequences of a certain feature of monic polynomials which forms the subject of the next section.

2. A property of polynomials

We turn our attention to a special family of monic polynomials with large positive coefficients. In the course of this section, we show that members of this family have all their roots in open left half of the complex plane and that the family includes almost all polynomials with large positive coefficients. In subsequent sections, this feature is used to investigate properties of linear high gain feedback controllers that stabilize a given system. Consider a monic polynomial

$$p_n(s) := s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \cdots + a_i s^{n-i} + \cdots + a_n, \tag{1}$$

whose coefficients are given by the special formula

$$a_i := k_i \alpha^{m(n,i)}, \quad i = 1, 2, \ldots, n, \tag{2}$$

where $\alpha, k_1, k_2, \ldots, k_n$ are strictly positive real numbers with $\alpha \to \infty$, and where the power $m(n,i)$ of α satisfies the recursion

$$\begin{cases} m(n,i) - m(n,i-1) = n - i + 1, \\ m(n,0) := 0. \end{cases} \tag{3}$$

The motivation behind considering this family of polynomials comes from an analysis of the effects of coefficient perturbations on a polynomial roots. The family of polynomials described by (1)–(3) exhibits a particularly transparent relationship in this regard, as shown later in the proof of Lemma 3. Furthermore, every monic polynomial can be represented in the form (1) with (2) and (3) by selecting proper values of the parameters $\alpha, k_1, k_2, \ldots, k_n$ (see (12) below); consequently, an examination of this family of polynomials carries general implications.

In preparation for studying the roots of this family of polynomials, we examine some of its basic features. First, note that, upon setting

$$a_0 := 1 \quad \text{and} \quad k_0 := 1,$$

the recursion among the powers is structured so that the ratio of consecutive coefficients satisfies the relation

$$\frac{a_i}{a_{i-1}} = \left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}, \quad i = 1, 2, \ldots, n.$$

To calculate the power of α from the recursion (3), we can use the arithmetic series formula $\sum_{i=1}^{n} i = n(n+1)/2$ to obtain

$$m(n, i) = \left[in - \frac{i(i-1)}{2}\right], \quad i = 1, 2, \ldots. \qquad (4)$$

Specifically, we have

$$m(n, 1) = n, \quad m(n, 2) = 2n - 1,$$
$$m(n, 3) = 3n - 3, \ldots, m(n, n-1) = \frac{(n-1)(n+2)}{2}, \quad m(n, n) = \frac{n(n+1)}{2}.$$

An inspection of (3) shows that $m(n, i) > m(n, i-1)$ for all $i = 1, \ldots, n$. Consequently, the powers of α increase from one coefficient of $p_n(s)$ to the next, as we move in the direction of declining powers of s. It is convenient at this point to look at some examples.

Example 1: For the case $n = 2$, we get $p_2(s) = s^2 + k_1\alpha^2 s + k_2\alpha^3$, where k_1 and k_2 can be any strictly positive numbers. For the case $n = 3$, we have $p_3(s) = s^3 + k_1\alpha^3 s^2 + k_2\alpha^5 s + k_3\alpha^6$, where k_1, k_2, k_3 can be any strictly positive real numbers.

Now, let $s_1, s_2, \ldots, s_n$ be the negatives of the roots of the polynomial (1). Then, we can write

$$p_n(s) = (s + s_1)(s + s_2)\ldots(s + s_n)$$
$$= s^n + k_1\alpha^n s^{n-1} + k_2\alpha^{2n-1}s^{n-2} + \cdots + k_n\alpha^{n(n+1)/2}.$$

To study the properties of the family of polynomials $\{p_n(s)\}$ as the parameter α approaches infinity, it is convenient to define the following relationship.

Definition 1: Two polynomials $A_1(\alpha)$ and $A_2(\alpha)$ in the variable α are α-equivalent (written $A_1(\alpha) =_{|\alpha} A_2(\alpha)$) if the highest powers of α in $A_1(\alpha)$ and in $A_2(\alpha)$ are equal and have the same coefficients.

Note that, when $A_1(\alpha)$ and $A_2(\alpha)$ are α-equivalent, then their ratio $A_1(\alpha)/A_2(\alpha) \to 1$ as $\alpha \to \infty$, i.e., α-equivalent expressions tend to become equal for large values of α. Consider now the quantities

$$s_i^0 := \left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}, \quad i = 1, 2, \ldots, n. \qquad (5)$$

Recalling that $s_1, s_2, \ldots, s_n$ are the negatives of the roots of the polynomial p_n(s), we intend to show that, with some mild exceptions,

$$\frac{s_i}{s_i^0} \to 1 \quad \text{as } \alpha \to \infty, \quad i = 1, 2, \ldots, n. \qquad (6)$$

In particular, this would indicate that, with some mild exceptions, polynomials of the form $p_n(s)$ have all their roots inside the open left half of the complex plane for large values of α. To this end, consider the following polynomial

$$q_n(s) := (s + s_1^0)(s^{n-1} + b_1 s^{n-2} + b_2 s^{n-3} + \cdots + b_{n-1})$$
$$= (s + k_1\alpha^n)(s^{n-1} + b_1 s^{n-2} + b_2 s^{n-3} + \cdots + b_{n-1}) \qquad (7)$$

where

$$b_i := \left(\frac{k_{i+1}}{k_1}\right)\alpha^{m(n-1,\, i)}, \quad i = 1, 2, \ldots, n-1, \; b_0 := 0. \quad (8)$$

More specifically,

$$b_1 = \left(\frac{k_2}{k_1}\right)\alpha^{n-1}, \quad b_2 = \left(\frac{k_3}{k_1}\right)\alpha^{2n-3},$$

and so on. For this choice of b_i, we show that the polynomial $q_n(s)$ is equivalent to the polynomial $p_n(s)$ for large values of α. To this end, denote by β_i the coefficient of s^{n-i} in the polynomial $q_n(s)$, i.e., set

$$q_n(s) = s^n + \beta_1 s^{n-1} + \cdots + \beta_n; \qquad (9)$$

then, the following is true.

Lemma 1:

$$\beta_i =_{|\alpha} a_i \text{ for all } i = 1, 2, \ldots, n.$$

Proof: A brief calculation shows that

$$\beta_i = b_i + b_{i-1}k_1\alpha^n = \left(\frac{k_{i+1}}{k_1}\right)\alpha^{m(n-1,\, i)} + \left(\frac{k_i}{k_1}\right)\alpha^{m(n-1,\, i-1)}k_1\alpha^n,$$
$$i = 1, 2, \ldots, n.$$

Using (4), we obtain

$$\beta_i = \frac{k_{i+1}}{k_1}\alpha^{[i(n-1)-(i(i-1)/2)]} + \frac{k_i}{k_1}\alpha^{[(i-1)(n-1)-((i-1)(i-2)/2)]}k_1\alpha^n$$
$$= \frac{k_{i+1}}{k_1}\alpha^{[i(n-1)-(i(i-1)/2)]} + \frac{k_i}{k_1}\alpha^{[in-(i(i-1)/2)]}, \; i = 1, 2, \ldots, n-1,$$

and

$$\beta_n = \alpha^n k_n \alpha^{m(n-1,\, n-1)}.$$

Using the fact that $k_1 = 1$ and simplifying, we get

$$\beta_i = \begin{cases} \begin{pmatrix} k_i\alpha^{[in-(i(i-1)/2)]} + k_{i+1}\alpha^{[i(n-1)-(i(i-1)/2)]}, \\ i = 1, 2, \ldots, n-1, \end{pmatrix} \\ k_n\alpha^{n(n+1)/2}, \qquad i = n. \end{cases} \tag{10}$$

In view of (2.4), we obtain $\beta_i =_{|\alpha} k_i\alpha^{m(n,\, i)}$, $i = 1, 2, \ldots, n$, so that, by (2), $\beta_i =_{|\alpha} a_i$, $i = 1, 2, \ldots, n$, and our proof concludes. □

Lemma 1 shows that the ratios between the coefficients of the polynomials $q_n(s)$ and $p_n(s)$ approach 1 as $\alpha \to \infty$. Qualitatively, this means that, for any bounded value of s, we have that $q_n(s)/p_n(s) \to 1$ as $\alpha \to \infty$. Referring to (7), we can define the polynomial $p'_{n-1}(s) := s^{n-1} + b_1 s^{n-2} + b_2 s^{n-3} + \cdots + b_{n-1}$, so that

$$q_n(s) = (s + s_1^0)p'_{n-1}. \tag{11}$$

Denoting the constants $k'_i := k_{i+1}/k_1$, we obtain from (8) that

$$b_i = k'_i\alpha^{m(n,\,i)} \quad i = 1, 2, \ldots, n,$$

so that $p'_{n-1}(s)$ has the same coefficient structure as the polynomial $p_{n-1}(s)$ of (1). As the coefficients k_i and k'_i represent arbitrary positive numbers, we can identify them with each other. This allows us to rewrite (11) in the form $q_n(s) = (s + s_1^0)p_{n-1}(s)$. Invoking Lemma 1, we obtain that $p_n(s) =_{|\alpha} (s + s_1^0)p_{n-1}(s)$. Iterating this process gives rise to the relationships

$$p_i(s) =_{|\alpha} (s + s_{n-i+1}^0)p_{i-1}(s), \quad i = 2, 3, \ldots, n,$$

where s_{n-i+1}^0 is given by (5). In qualitative terms, these relationships imply that the roots of the polynomial $p_n(s)$ tend toward the values $-s_1^0, -s_2^0, \ldots, -s_n^0$ as $\alpha \to \infty$. Stating this fact in accurate form, finding its ramifications, and proving its validity are the main objectives of the remaining part of this section.

Note that, since $s_i^0 > 0$ for all $i = 1, 2, \ldots, n$, the result mentioned in the previous paragraph would yield a family of monic polynomials whose roots are all in the open left half of the complex plane. Clearly, when any of these polynomials is assigned as the characteristic polynomial of a linear system, the resulting system is asymptotically stable for large values of α. This observation leads us to the derivation of a rather large family of stabilizing feedback compensators discussed in subsequent sections of the paper.

Before continuing, we note that the family of polynomials $\{p_n(s)\}$ of (1) and (2) is, in fact, a large family of polynomials. Indeed, let P^+ denote the set of all monic polynomials with positive coefficients, and consider a member $p(s)$ of P^+, say

$$p(s) = s^n + c_1 s^{n-1} + \cdots + c_n,$$

where $c_1, c_2, \ldots, c_n > 0$. We can bring $p(s)$ into a polynomial with coefficients given by (2) by setting $k_1 := 1$ and determining the remaining constants α, k_2, $k_3, \ldots, k_n > 0$ from the equations

$$k_i\alpha^{m(n,\,i)} = c_i, \quad i = 1, 2, \ldots, n, \tag{12}$$

where $m(n, i)$ is given by (3). Setting $k_1 := 1$ has the effect of normalizing the values of the remaining coefficients.

Example 2: Consider the polynomial $p(s) = s^3 + c_1 s^2 + c_2 s + c_3$, where $c_1, c_2, c_3 > 0$. To bring the coefficients of this polynomial into the form (2), we obtain the equations $k_1\alpha^3 = c_1$, $k_2\alpha^5 = c_2$, $k_3\alpha^6 = c_3$. Setting $k_1 := 1$, yields $\alpha = \sqrt[3]{c_1}$, $k_2 = c_2/\alpha^5$, and $k_3 = c_3/\alpha^6$.

Needless to say, not all monic polynomials with positive coefficients have roots that approach the values (5); a restriction on the coefficients $c_1, c_2, \ldots, c_n$ of $p(s)$ is required in order for that to happen as $\alpha \to \infty$. Nevertheless, as our forthcoming discussion indicates, this restriction excludes only a small subset of monic polynomials with large positive coefficients. First, some introductory notions.

Let $v = (v_1, v_2, \ldots, v_n)$ and $v' = (v'_1, v'_2, \ldots, v'_n)$ be two vectors in R^n. Using the Euclidean norm $|v| = \left(\sum_{i=1}^n v_i^2\right)^{1/2}$ and the inner product $v \cdot v' := \sum_{i=1}^n v_i v'_i$, the angle ϑ between the two vectors v and v' is often defined by

$$\vartheta := \cos^{-1}\frac{v \cdot v'}{|v||v'|}.$$

Now, for a real number $\gamma > 0$, let $\chi(\gamma)$ be the set of all vectors $w = (w_1, w_2, \ldots, w_n) \in R^n$ satisfying the following two conditions:

(i) $w_1 > 0$, and
(ii) $\gamma \leq (w_{i+1}/w_i) \leq 1/\gamma$ for all $i = 1, 2, \ldots, n-1$.

Also, let R^{+n} be the set of vectors in R^n having strictly positive coordinate values; i.e., R^{+n} consists of all vectors $(x_1, x_2, \ldots, x_n) \in R^n$ for which $x_i > 0$ for all $i = 1, 2, \ldots, n$. We refer to R^{+n} as a *cone*. The term 'cone' is used here in a generalized sense, meaning an n-dimensional domain created by the motion of a straight ray whose origin is attached to a fixed point. Applying this definition to the set $\chi(\gamma)$, it follows that $\chi(\gamma)$ forms an infinite cone within R^{+n}. The vertex angle of $\chi(\gamma)$ approaches $\pi/2$ as $\gamma \to 0$.

Definition 2: A *horn* of a cone is the open end of the cone: it is the set of all vectors of the cone whose length exceeds r, where r is a positive real number.

Note that the value of r in the definition is unspecified and may vary from case to case; when using the term "horn" in the sequel, we shall consider r to be arbitrarily large. In intuitive terms, a horn is the open infinite end of a cone.

Let $\chi(\gamma, M)$ be the set of all vectors in $\chi(\gamma)$ whose length exceeds M, i.e., all $w \in \chi(\gamma)$ satisfying $|w| \geq M$. Then, the set $\chi(\gamma, M)$ forms a horn of the cone $\chi(\gamma)$.

Example 3: For $n = 2$, the set $\chi(\gamma, M)$ takes the form of figure 1.

We apply now these notions to a monic polynomial $p(s) = s^n + c_1 s^{n-1} + \cdots + c_n$ of degree n with positive coefficients. First, define the ratios between consecutive coefficients

$$\gamma_i := \frac{c_i}{c_{i-1}}, \quad i = 1, 2, \ldots, n \quad \text{where } c_0 := 1,$$

and refer to $\gamma_1, \gamma_2, \ldots, \gamma_n$ as the *slopes* of the polynomial $p(s)$. Each set of slopes $\gamma_1, \gamma_2, \ldots, \gamma_n$ can be considered as a point in the n dimensional space R^n. In this space, we define the multivariable polynomials

$$\phi_i(\gamma_1, \gamma_2, \ldots, \gamma_n) := (n-1)(-\gamma_i)^{n-2} + (n-2)\gamma_2(-\gamma_i)^{n-3} + \cdots + \gamma_2 \cdots \gamma_{n-1},$$

where $i = 1, \ldots, n$ and $n \geq 3$. For each $i \in \{1, \ldots, n\}$, let $\Phi_i(n)$ be the surface that consists of all slopes $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in R^n$ that satisfy the equation

$$\phi_i(\gamma_1, \gamma_2, \ldots, \gamma_n) = 0. \qquad (13)$$

Note that the surface $\Phi_i(n)$ is of measure zero with respect to the standard Lebesgue measure in R^n for all $i = 1, 2, \ldots, n$. Consequently, the set

$$\Phi(n) := \cup_{i=1,2,\ldots,n} \Phi_i(n),$$

being the union of a finite number of sets of measure zero, has the following property.

Lemma 2: *$\Phi(n)$ is a set of measure zero in R^n (with respect to the standard Lebesgue measure).*

Example 4: For $n = 3$, we obtain

$$\Phi_1\colon 2(-\gamma_1) + \gamma_2 = 0, \text{ or } \gamma_2 = 2\gamma_1;$$
$$\Phi_2\colon 2(-\gamma_2) + \gamma_2 = 0, \text{ or } \gamma_2 = 0;$$
$$\Phi_3\colon 2(-\gamma_3) + \gamma_2 = 0 \text{ or } \gamma_2 = 2\gamma_3$$

As we can see, in this case, all three surfaces are planes. They can be drawn as in figure 2.

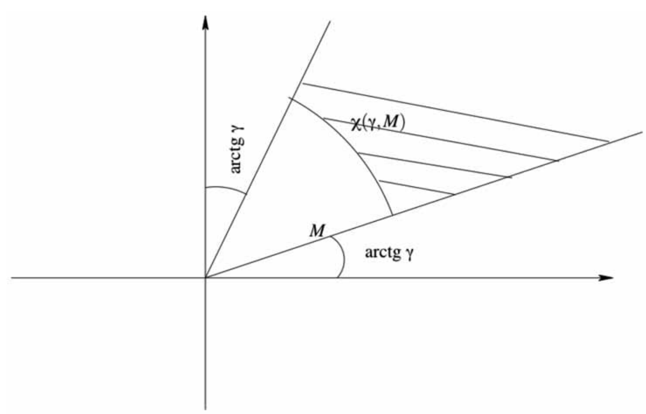

Figure 1. A cone.

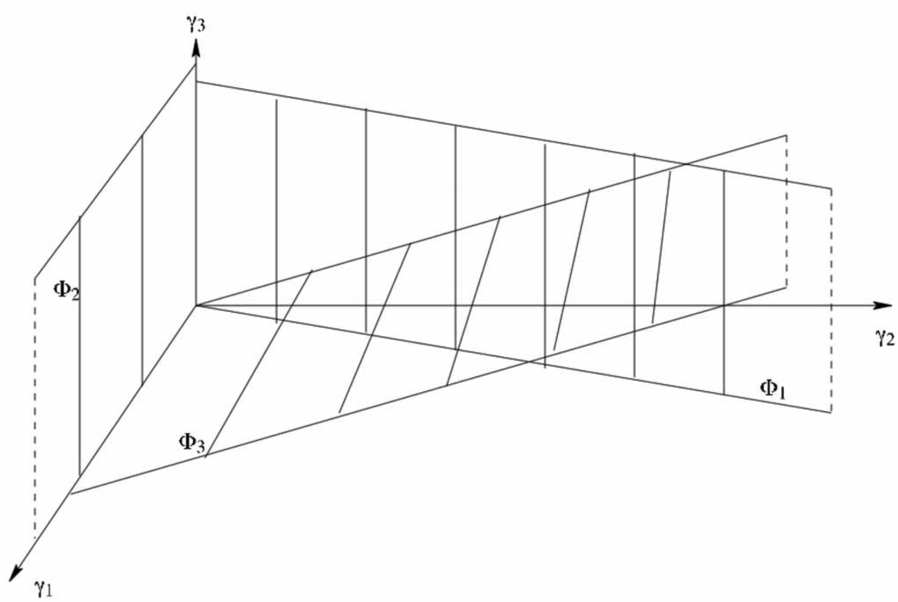

Figure 2. The set $\Phi(n)$ for $n = 3$.

Now, for a real number $\delta > 0$, build around the surface $\Phi_i(n)$ a domain $S_i(n, \delta)$ in R^n given by

$$S_i(n,\delta) := \{\gamma_1, \gamma_2, \ldots, \gamma_n \in R^n : |\phi_i(\gamma_1, \gamma_2, \ldots, \gamma_n| \le \delta\},$$
$$i = 1, 2, \ldots, n.$$

Then, define the set

$$S(n,\delta) := \begin{cases} \cup_{i=1,2,\ldots,n} S_i(n,\delta), & \text{for } n \ge 3, \\ \emptyset & \text{for } n = 1, 2, \end{cases} \tag{14}$$

where Ø denotes the empty set. Finally, consider the difference set

$$V(\delta, M) := \chi(\delta, M) \backslash S(n, \delta), \tag{15}$$

where M, $\delta > 0$. Then, it follows from Lemma 2 that, as $\delta \to 0$, the domain $V(\delta, M)$ includes almost all vectors in R^n having positive components and length exceeding M. In the limit, as $\delta \to 0$, the only vectors excluded from the set are those whose slopes satisfy one of the relations (13). Intuitively, this means that, as $\delta \to 0$, a random selection of an n dimensional vector of length M or longer yields, with probability one, a vector in the set $V(\delta, M)$ (assuming all slopes are equally probable).

The following statement indicates that, for sufficiently large M, a polynomial with coefficients in $V(\delta, M)$ has roots that are approximately given by (5). Consequently, all roots of such a polynomial are located in the open left half of the complex plane. Clearly, assigning such a polynomial as the characteristic polynomial of a linear system yields an asymptotically stable system.

Lemma 3: *Let $-s_1, -s_2, \ldots, -s_n$ be the roots of the monic polynomial $p(s) = s^n + c_1 s^{n-1} + c_2 s^{n-2} + \cdots + c_n$, and let $V(\delta, M)$ be given by* (15). *Then, for every pair of real numbers ε, $\delta > 0$, there is a real number $M > 0$ such that, for all coefficients $(c_1, c_2, \ldots, c_n) \in V(\delta, M)$, the roots of $p(s)$ satisfy*

$$\left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}(1-\varepsilon) \le s_i \le \left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}(1+\varepsilon),$$
$$i = 1, 2, \ldots, n,$$

where $k_0 := 1$, $k_1 := 1$, and $k_2, k_3, \ldots, k_n$, and α are determined by the equation (12).

The proof of Lemma 3 is listed below. In intuitive terms, Lemma 3 originates from the fact that the polynomial $p(s)$ stays close to the polynomial $q_n(s)$ of (9) in the limit as $\alpha \to \infty$. The Lemma indicates that almost all monic polynomials of degree n with sufficiently large positive coefficients have roots that are approximately equal to the ones given by (5). In particular, this implies that all such polynomials have their roots inside the open left half of the complex plane. This point underlies much of our discussion in the following sections. It is convenient to introduce the following notion (all measures are with respect to the standard Lebesgue measure in R^n).

Definition 3: Let $\Gamma(\delta) \subset R^n$ be a family of subsets depending on a parameter $\delta > 0$. Then, $\Gamma(\delta)$ is a *virtual horn* if the following conditions hold:

(a) There is a horn $H \subset R^n$ such that the difference set $H \backslash \Gamma(\delta)$ approaches a set of measure zero as $\delta \to 0$.
(b) $\Gamma(\delta)$ is the union of a finite number m of simply connected sets, where m is independent of δ.

When (i) and (ii) are valid, we say that $\Gamma(\delta)$ is *virtually equal* to the horn H. If G is a subhorn of H, then we say that $\Gamma(\delta)$ *virtually includes* the horn G.

Note that the set $V(\delta, M)$ of (15) forms a virtual horn; it is virtually equal to the horn R^{+n} of all vectors with positive coordinates in R^n. In these terms, Lemma 3 can be restated in the following somewhat briefer form.

Corollary 1: *The set of real coefficients $(c_1, c_2, \ldots, c_n)$ for which the monic polynomial $p(s) = s^n + c_1 s^{n-1} + \cdots + c_n$ has all its roots inside the open left half of the complex plane virtually includes the horn R^{+n}.*

We turn now to the proof of the Lemma 3.

Proof of Lemma 3: In view of (12), we can assume without loss of generality that the monic polynomial $p(s)$ has coefficients of the form (2). The roots of our polynomial are, of course, determined by the equation

$$p(s) = s^n + a_1 s^{n-1} + \cdots + a_n = 0, \tag{16}$$

where we set $c_i = a_i$, $i = 1, 2, \ldots, n$, with a_i being given by (2). Recall the polynomial $q_n(s) = s^n + \beta_1 s^{n-1} + \cdots + \beta_n$ of (9), whose coefficients are given by (10). The discrepancy between the coefficients of the polynomials $p(s)$ and $q_n(s)$ is given by

$$\Delta a_j := \beta_j - a_j = k_{j+1}\alpha^{[j(n-1)-(j(j-1)/2)]}, \quad j = 1, 2, \ldots, n. \tag{17}$$

Now, by the construction in (7), the first root of $q_n(s)$ is $-s_1^0$ where s_1^0 is given by (5). The discrepancies $\Delta a_1, \Delta a_2, \ldots, \Delta a_n$ between the coefficients of the two polynomials cause the root $-s_1$ of $p(s)$ to deviate from the root s_1^0 of $q_n(s)$. Of course, the same discrepancies between the coefficients will also cause each one of the roots $-s_i$ of $p(s)$ to deviate from the value of the corresponding root s_i^0 of $q_n(s)$. Our proof will be complete upon showing that these root deviations satisfy the relations stated in the lemma. To derive a bound on the root deviations, we use the Taylor series first order error bound.

Write

$$s_i = s_i^0(1+\varepsilon_i), \tag{18}$$

where ε_i describes the "fractional" impact of the deviations $\Delta a_1, \dots, \Delta a_n$ on root i. Considering ε_i as a function of the coefficients $a_1, a_2, \dots, a_n$, we can write $\partial s_i/\partial a_j = s_i^0(\partial\varepsilon_i/\partial a_j)$, or

$$\frac{\partial s_i}{\partial a_j} = \frac{1}{s_i^0}\left(\frac{\partial s_i}{\partial a_j}\right) \tag{19}$$

To obtain the derivative $\partial s_i/\partial a_j$, we can differentiate equation (16) which characterizes the roots. This yields

$$\begin{aligned}&\frac{\partial}{\partial a_j}[s_i^n + a_1 s_i^{n-1} + \cdots + a_n]\\ &= n s_i^{n-1}\frac{\partial s_i}{\partial a_j} + a_1(n-1)s_i^{n-2}\frac{\partial s_i}{\partial a_j} + \cdots + s_i^{n-j}\\ &\quad + a_j(n-j)s_i^{n-j-1}\frac{\partial s_i}{\partial a_j} + \cdots + a_{n-1}\frac{\partial s_i}{\partial a_j} = 0,\end{aligned}$$

so that

$$\frac{\partial s_i}{\partial a_j} = \frac{-s^{n-j}}{ns^{n-1} + a_1(n-1)s^{n-2} + a_2(n-2)s^{n-3} + \cdots + a_{n-1}}.$$

Evaluating the derivative at the point $s = -s_i^0$ of (5) and assuming that the denominator is not zero, we obtain (20)

Let $\Delta\varepsilon_{ij}$ be the contribution of the discrepancy Δa_j to the deviation ε_i. Then, using the first order Taylor series error bound together with (17) and (19), we obtain (21) To obtain better insight, we express the last formula in terms of the slopes of the coefficient vector, as follows. Define the ith coefficient slope

$$\gamma_i := \frac{a_i}{a_{i-1}}, \quad i = 1, 2, \dots, n \quad \text{where } a_0 := 1.$$

At the nominal coefficients (2), we obtain $\gamma_i = (k_i/k_{i-1})\alpha^{n-i+1}$, so that, by (5), we have $\gamma_i = s_i^0,\ i = 1, 2, \dots, n$. Substituting into (21), yields (22) where we assume that the denominator is not zero, i.e., that no roots of the denominator polynomial are among the slopes $\gamma_1, \gamma_2, \dots, \gamma_n$. Further, note that

$$\begin{aligned}\gamma_{j+1}\gamma_j\dots\gamma_2 &= \frac{k_{j+1}}{k_j}\frac{k_j}{k_{j-1}}\dots\frac{k_2}{k_1}\alpha^{n-j}\alpha^{n-j+1}\alpha^{n-j+2}\dots\alpha^{n-1}\\ &= k_{j+1}\alpha^{\sum_{i=0}^{j-1}(n-j+i)} = k_{j+1}\alpha^{[j(n-j)+\sum_{i=0}^{j-1} i]}\\ &= k_{j+1}\alpha^{[j(n-j)+(j(j-1)/2)]} = k_{j+1}\alpha^{[jn-(j(j+1)/2)]}.\end{aligned}$$

Also, since $m(n-1,j) - [jn-(j(j+1)/2)] = j(n-1) - (j(j-1)/2) - [jn-(j(j+1)/2)] = 0$ we obtain

$$\frac{\partial s_i}{\partial a_j} = \frac{-\left(-\left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-j+1}\right)^{n-j}}{\left[\left(n\left(-\left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}\right)^{n-1} + k_1\alpha^n(n-1)\left(-\left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}\right)^{n-2} + k_2\alpha^{2n-1}(n-2)\left(-\left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}\right)^{n-3} + \cdots + k_{n-1}\alpha^{(n-1)n-((n-1)(n-2)/2)}\right)\right]}. \tag{20}$$

$$\begin{aligned}|\Delta\varepsilon_{ij}| &\le \sup\left|\frac{\partial\varepsilon_i}{\partial a_j}\Delta a_j\right| = \sup\left|\frac{1}{s_i^0}\frac{\partial s_i}{\partial a_j}\Delta a_j\right|\\ &= \sup\left|\frac{\left(-\left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}\right)^{n-j-1}\left(\frac{k_{j+1}}{k_1}\right)\alpha^{j(n-1)-(j(j-1)/2)}}{\left[\left(n\left(-\left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}\right)^{n-1} + k_1\alpha^n(n-1)\left(-\left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}\right)^{n-2} + k_2\alpha^{2n-1}(n-2)\left(-\left(\frac{k_i}{k_{i-1}}\right)\alpha^{n-i+1}\right)^{n-3} + \cdots + k_{n-1}\alpha^{(n-1)n-((n-1)(n-2)/2)}\right)\right]}\right|.\end{aligned} \tag{21}$$

$$|\Delta\varepsilon_{ij}| \le \sup\left|\frac{\gamma_i^{n-j-1}}{\left(n(-\gamma_i)^{n-1} + (n-1)\gamma_1(-\gamma_i)^{n-2} + (n-2)\gamma_1\gamma_2(-\gamma_i)^{n-3} + \cdots + \gamma_1\gamma_2\dots\gamma_{n-1}\right)}\left(k_{j+1}\alpha^{m(n-1,j)}\right)\right|, \tag{22}$$

that $k_{j+1}\alpha^{m(n-1,j)} = \gamma_{j+1}\gamma_j \dots \gamma_2$. Substituting into (22) yields

$$\left|\Delta\varepsilon_{ij}\right| \leq \sup\left|\frac{\gamma_i^{n-j-1}\gamma_{j+1}\gamma_j\dots\gamma_2}{\left(n(-\gamma_i)^{n-1}+(n-1)\alpha^n(-\gamma_i)^{n-2}+(n-2)\alpha^n\gamma_2(-\gamma_i)^{n-3}+\dots+\alpha^n\gamma_2\dots\gamma_{n-1}\right)}\right| \tag{23}$$

for all $n \geq 3$; for $n = 2$, the last term in the denominator is $\alpha^n\gamma_2$, since $\gamma_1 = \alpha^n$ was already inserted earlier. Consequently, for fixed slopes $\gamma_1, \gamma_2, \dots, \gamma_n$, we have $|\Delta\varepsilon_{ij}| \to 0$ when $\alpha \to \infty$, as long as none of the slopes γ_1, $\gamma_2, \dots, \gamma_n$ is a root of the denominator.

Recall that, by the definition of the set $\chi(\delta, M)$, we have $\delta \leq \gamma_i \leq (1/\delta)$ for all $i = 1, 2, \dots, n$, where we have assumed (without loss of generality) that $\delta \leq (1/\delta)$. Also, recalling (14), we obtain from (23) that, for all slopes $\gamma_1, \gamma_2, \dots, \gamma_n \notin S(n, \delta)$ and for sufficiently large α, we can write

$$\left|\Delta\varepsilon_{ij}\right| \leq \left|\frac{(1/\delta)^{n-1}}{\alpha^n\delta - n\delta^{n-1}}\right| = \left|\frac{1}{\alpha^n\delta^n - n\delta^{2(n-1)}}\right|$$

for all $i, j = 1, 2, \dots, n$. Taking into account the contributions of the deviations in all n coefficients, the total relative error ε_i in the root s_i satisfies $|\varepsilon_i| \leq \sum_{j=1}^{n} |\Delta\varepsilon_{ij}| \leq n/|\alpha^n\delta^n - n\delta^{2(n-1)}|$. The last expression is clearly independent of the index i; setting $\varepsilon := \max_{i=1,\dots,n} |\varepsilon_i|$ we can rewrite the last inequality in the form $\varepsilon \leq n/|\alpha^n\delta^n - n\delta^{2(n-1)}|$. Finally, taking $\alpha := N$, we conclude that our lemma is valid for all real numbers $M \geq N$, where N satisfies the inequality $1/|N^n\delta^n - n\delta^{2(n-1)}| = \varepsilon/n$. This completes the proof. □

Furthermore, it is well known that all monic polynomials whose roots are in the open left half of the complex plane must have positive coefficients. Combining this with Corollary 1 yields the following conclusion, which summarizes the discussion of this section.

Theorem 1: *The set of real coefficients* $(c_1, c_2, \dots, c_n)$ *for which the monic polynomial* $p(s) = s^n + c_1 s^{n-1} + \dots + c_n$ *has all its roots inside the open left half of the complex plane is virtually equal to the horn* R^{+n}.

3. State feedback and tolerance horns

We consider now the implications of Lemma 3 on the tolerance allowed in the implementation of stabilizing static state feedback controllers. To that end, consider a linear time-invariant input/state system, namely, a system described by the realization

$$\Sigma : \quad x(t) = Ax(t) + Bu(t). \tag{24}$$

Here, A and B are constant matrices. Let n be the dimension of the state vector x and let m be the dimension of the input vector u. Assume that the realization is reachable, and apply a static state feedback around the system Σ (figure 3). This results in the input

$$u(t) = v(t) - Kx(t),$$

where K is an $m \times n$ constant matrix of real numbers and $v(t)$ is the external input function. In the diagram, Σ represents the input/state system (24). The state feedback matrix K is selected so as to stabilize the system Σ, and the objective is to examine the tolerance allowed in the implementation of K. We concentrate on the case of high feedback gains, i.e., the case when the matrix K has entries with large absolute values. First, some preliminary notions, starting with the familiar notion of a circular cone.

Definition 4: Let $v \in R^n$ be a nonzero vector, and let $0 \leq \rho \leq \pi$ be a real number. The *circular cone* $\Xi(v, \rho)$ with *vertex angle* ρ around the vector v is the set of all vectors $w \in R^n$ satisfying

$$\Xi(v, \rho) := \left\{ w \neq 0 \in R^n : \cos^{-1}\frac{v \cdot w}{|v||w|} \leq \rho \right\}.$$

Let $S \subset R^n$ be a set containing a straight ray. The *inner span* ρ of S is the vertex angle of the largest circular cone that can fit into S, i.e., $\rho := \sup\{\varphi : \Xi(v, \varphi) \subset S, v \in S\}$. The inner span of a horn is the inner span of the cone generating the horn.

In the special case when S is a circular cone, the inner span is the same as the vertex angle of S. It is interesting to look at the inner span of a matrix image of R^{+n}.

Definition 5: Let Q be a real matrix with n columns, and let $Q[R^{+n}]$ be the image of R^{+n} through the matrix Q. The *column inner span* of Q is the inner span of the

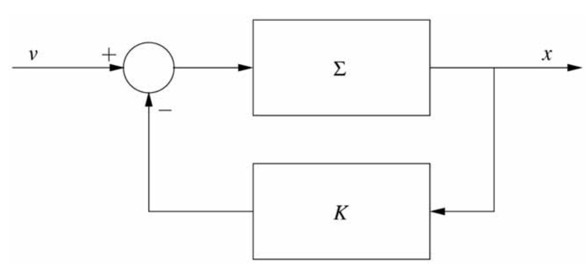

Figure 3. State feedback.

set $Q[R^{+n}]$. The *row inner span* of a matrix is the column inner span of its transpose.

Example 5: Consider the matrix

$$Q = \begin{pmatrix} 1-\varepsilon & 1 \\ 1 & 1 \end{pmatrix},$$

where $0 < \varepsilon < 1$. The image set $Q[R^{+2}]$ is then the set of all linear combinations with positive coefficients of the two vectors $(1-\varepsilon,\ 1)^T$, $(1,\ 1)^T$, i.e., the set of vectors $(a(1-\varepsilon)+b,\ a+b)^T$, where $a,\ b > 0$. Using the parallelogram rule, it can be seen that the column inner span θ of the matrix Q is given by half of the angle between the two columns of Q, i.e., by

$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{2-\varepsilon}{\sqrt{2(2-2\varepsilon+\varepsilon^2)}}\right).$$

As we can see, the angle θ approaches 0 as $\varepsilon \to 0$ i.e., as the matrix Q approaches singularity.

Consider the reachable linear input/state system of (24), and let $C' = (B, AB, A^2B, \ldots, A^{n-1}B)$ be its controllability matrix. As the realization is reachable, we have

$$\operatorname{rank} C' = n,$$

where n is the dimension of the state. When the system Σ has more than one input variable, the matrix C' is not a square matrix. The following process (see also Brunovski (1970) and Popov (1972)) extracts from C' an $n \times n$ non-singular matrix C, called the *reduced controllability matrix*.

Algorithm 1: Derivation of the reduced controllability matrix of the realization (24).

Step 1: Let $B_1, B_2, \ldots, B_m$ be the columns of the input matrix B. Define the $n \times n$ sub-matrices

$$C'_i := (B_i, AB_i, \ldots, A^{n-1}B_i), \quad i = 1, \ldots, m.$$

set $k := 1$.

Step 2: Derive a list of integers by the following recursive process.

(a) First, set

$$n_1 := \max_{i=1,\ldots,m} \operatorname{rank} C'_i,$$

and let i_1 be an integer for which $\operatorname{rank} C'_{i_1} = n_1$. If $n_1 = n$, then go to Step 3; otherwise continue to (b).

(b) Using recursion, assume that the integers $n_1,\ n_2, \ldots, n_k$ and $i_1, i_2, \ldots, i_k$ have been derived; define

$$n'_{k+1} := \max_{i=1,\ldots,m} \operatorname{rank}\left(C'_{i_1}, C'_{i_2}, \ldots, C'_{i_k}, C'_i\right),$$

and let $i_{k+1} \in \{1, \ldots, m\}$ be an integer for which $\operatorname{rank}(C'_{i_1}, C'_{i_2}, \ldots, C'_{i_k}, C'_{i_{k+1}}) = n'_{k+1}$. Set $n_{k+1} := n'_{k+1} - (n_1 + n_2 + \cdots + n_k)$.

(c) If $n_1 + n_2 + \cdots + n_{k+1} = n$, then go to Step 3; otherwise, repeat Step 2 with the value of k increased by 1.

Step 3: Define the matrices $C_{i_j} := (B_{i_j}, AB_{i_j}, \ldots, A^{n_j-1}B_{i_j})$, $j = 1, 2, \ldots, k$. The *reduced controllability matrix* is then given by

$$C := \left(C_{i_1}, C_{i_2}, \ldots, C_{i_k}\right).$$

Note that the reduced controllability matrix of a reachable realization is invertible by construction. Consequently, it can be used to induce a similarity transformation on the realization (24); defining the matrices

$$A' := C^{-1}AC,\ B' := C^{-1}B,$$

we obtain the so-called *controllability canonical form* realization

$$\dot{x} = A'x + B'u. \tag{25}$$

It can be shown from our construction that the reduced controllability matrix of the controllability canonical form is the identity matrix.

Now, let T_p be a similarity transformation that takes the realization (25) into the controller form (see Popov (1972) for a construction of the matrix T_p). Applying the similarity transformation T_p to the realization (25), we obtain the realization

$$\dot{z} = A_c z + B_c u, \tag{26}$$

where the matrices $A_c := T_p^{-1}A'T_p$ and $B_c := T_p^{-1}B'$ are of the form

$$A_c = \begin{pmatrix} (A_1) & \begin{pmatrix} a_{1,1} \\ 0 \end{pmatrix} & \begin{pmatrix} a_{1,2} \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} a_{1,k} \\ 0 \end{pmatrix} \\ (0) & (A_2) & \begin{pmatrix} a_{2,1} \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} a_{2,k-1} \\ 0 \end{pmatrix} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (0) & (0) & \cdots & (0) & (A_k) \end{pmatrix},$$

$$B_c = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}.$$

Here, the blocks $A_1,\ A_2, \ldots, A_k$ on the main diagonal are matrices in companion form; the dimension of the matrix A_j is $i_j \times i_j$, and $a_{i,j}$ are scalars. In fact, (26) is a combination of single input systems in the controller form, where each single input system is given by

$$\Sigma_j: \quad \dot{x}^j = A_j x^j + b_j u^j, \tag{27}$$

where

$$A_j = \begin{pmatrix} -a_j^1 & -a_j^2 & \cdots & -a_j^{i_j-1} & -a_j^{i_j} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad b_j = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix},$$

$$j = 1, \dots, k. \tag{28}$$

Furthermore, it can be verified that the similarity transformation is given by $T_p = C_c^{-1}$, where C_c is the controllability matrix of the controller form realization (26). The matrix C_c is composed of the controllability matrices of the sub-realizations $\Sigma_1, \dots, \Sigma_k$ of (26). Let C_j be the controllability matrix of the sub-realization (27); being the controllability matrix of a single-input realization in controller form, C_j is in the lower triangular form

$$C_j = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & * \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix},$$

where all elements on the main diagonal are 1 (and * represents an unspecified entry). A direct examination shows then that

$$C_c = \begin{pmatrix} C_1 & * & \cdots & * \\ 0 & C_2 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & C_k \end{pmatrix}.$$

In other words, C_c is a lower triangular matrix with ones on its main diagonal, so that $\det C_c = 1$. Thus, the matrix C_c never gets close to being singular.

Example 6: Consider the case of (26) with $k = 1$ and $i_1 = 3$. Then,

$$A_c = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

so the controllability matrix is

$$C_c = \begin{pmatrix} 1 & -a_1 & a_1^2 - a_2 \\ 0 & 1 & -a_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consequently, we always have $\det C_c = 1$, irrespective of the values of a_1, a_2, and a_3.

The combined similarity transformation

$$C_N := C C_c^{-1} \tag{29}$$

takes the original realization (24) to the multivariable controller form (26).

Definition 6: The matrix $C_N = C C_c^{-1}$ is the *normalized controllability matrix* of the realization (24).

As we have seen, the normalized controllability matrix is the similarity transformation that takes a realization into its controller canonical form. It plays a critical role in determining the tolerance allowed for the parameters of a stabilizing state feedback controller, as we now explain. Consider first the stabilization of an input/state system in the multivariable controller form (26). To this end, apply the static state feedback

$$K_{c,j} = \left(k_{c,j}^1, k_{c,j}^2, \dots, k_{c,j}^{i_j}\right) \tag{30}$$

to subsystem j of (27), $j = 1, \dots, k$. When these feedback vectors are combined into one static state feedback matrix, we obtain the feedback matrix represented by the direct sum

$$K_c = K_{c,1} \oplus K_{c,2} \oplus \cdots \oplus K_{c,k},$$

i.e., by the matrix

$$K_c = \begin{pmatrix} K_{c,1} & 0 & 0 & \cdots & 0 \\ 0 & K_{c,2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & K_{c,k} \end{pmatrix}. \tag{31}$$

Now, let $\mathcal{K}_c$ be the class of static state feedback matrices that stabilize the system (26). Clearly, the class $\mathcal{K}_c$ depends on the values of the entries $a_j^1, \dots, a_j^{i_j}$ of (28), $j = 1, 2, \dots, k$. It is convenient to consider first the case where all these entries are equal to zero, i.e., the case where

$$a_j^i = 0, i = 1, \dots, i_j, \ j = 1, \dots, k. \tag{32}$$

Denote by Σ_j^0 the instance of Σ_j satisfying (32). Applying the feedback (30) to Σ_j^0, we obtain the transition matrix

$$A^0(K_{c,j}) = \begin{pmatrix} -k_j^1 & -k_j^2 & -k_j^3 & \cdots & -k_j^{i_j} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

whose characteristic polynomial is

$$a_j^0(s) := s^{i_j} + k_j^1 s^{i_j - 1} + \cdots + k_j^{i_j}.$$

Let $S^0(i_j)$ be the set of all vectors $K_{c,j}$ for which the polynomial $a_j^0(s)$ has all its roots in the open left half of the complex plane, i.e., $S^0(i_j)$ describes the set of all stabilizing feedback compensators for this subsystem. In view of Lemma 3, the set $S^0(i_j)$ includes the virtual horn $V(\delta, M)$ of (15). The set of all stabilizing feedback vectors for the system (27) is then given by

$$S_c(i_j) := S^0(i_j) + (a_j^1, a_j^2, a_j^3, \dots, a_j^{i_j}).$$

In view of (31), this yields the set of stabilizing static state feedback compensators for the system (26) in the form

$$S_c := S_c(i_1) \oplus S_c(i_2) \oplus \cdots \oplus S_c(i_k).$$

Recalling the similarity transformation C_N of (29), it follows by a direct calculation that the static state feedback compensator

$$K := K_c C_N \tag{33}$$

stabilizes the original realization (24), with K_c being any static state feedback that stabilizes the controller form (26). Consequently, the set of static state feedback compensators

$$S := S_c C_N$$

forms a set of stabilizing static state feedback matrices for the original realization (24).

In view of Corollary 1, the set of row vectors $S_c(i_j)$ virtually includes the horn $(R^{+i_j})^T$, where $(R^{+i_j})^T$ denotes the set of all row vectors of dimension i_j with positive coefficients. Let C_N^1 be the submatrix of C_N that consists of the top-left $i_1 \times i_1$ block; let C_N^2 be the submatrix of C_N consisting of the next $i_2 \times i_2$ block along the main diagonal, and so on. Let

$$C^+(j) := \left[R^{+i_j}\right]^T C_N^j$$

be the horn spanned by the rows of the matrix C_N^j, $i = 1, 2, \dots, k$. Applying Theorem 1, we obtain the following.

Theorem 2: *The set of stabilizing static state feedback controllers of the system Σ is virtually equal the sum of horns*

$$C^+ := C^+(1) \oplus C^+(2) \cdots \oplus C^+(k). \tag{34}$$

Further, let ρ_i be the inner row span of the matrix C_N^i, $i = 1, 2, \dots, k$. It follows then from (31) and (34) that the non-zero entries of the class of stabilizing static state feedback controllers for the system Σ virtually include a horn with the span

$$\rho := \min_{i=1,2,\dots,k} \rho_i.$$

The number ρ provides an overall indication of the tolerance available in the implementation of high-gain stabilizing state feedback controllers for the system Σ. We refer to ρ as the *controllability inner span* of the system Σ. The discussion so far can be summarized in the following brief statement.

Corollary 2: *Let Σ be a reachable input/state system, let ρ be the controllability inner span of Σ, and let S be the family of stabilizing static state feedback controllers of Σ. Then, the non-zero entries of the family S virtually include a horn with row span equal to ρ.*

In intuitive terms, the combination of Theorem 2 and Corollary 2 indicates that the set of high-gain stabilizing state feedback controllers is almost equal to the horn spanned by the normalized controllability matrix C_N. The closer C_N is to being singular, the less tolerance is available when implementing stabilizing state feedback controllers. This fact brings to light an important connection between the implementation tolerance and features of the normalized controllability matrix. We conclude this section with a simple example.

Example 7: Consider the realization $\dot{x} = Ax + Bu$, where x is of dimension 2. Assume that this realization has the normalized controllability matrix

$$C_N = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and that, after applying the similarity transformation C_N, we obtain the controller form

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

Using a negative state feedback configuration with the state feedback vector $k_c = (a, b)$, we obtain the realization

$$\dot{z} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

The characteristic polynomial is given by $a(s) := s^2 + as + b$. Both roots of this polynomial have negative real parts whenever $a, b > 0$ (there is no need to consider horns in this case). Thus, the set of stabilizing feedback vectors $\{K_c\}$ covers the entire first quadrant of our feedback vector space, as depicted in figure 4. (In this case, there is no need to restrict to high gain feedback.)

Now, let us transform the feedback back to the original coordinate system. Denoting by $K := (\alpha, \beta)$ the feedback vector in the original realization, we obtain from (33) that

$$K = (\alpha, \beta) = K_c C_N = (a, b) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

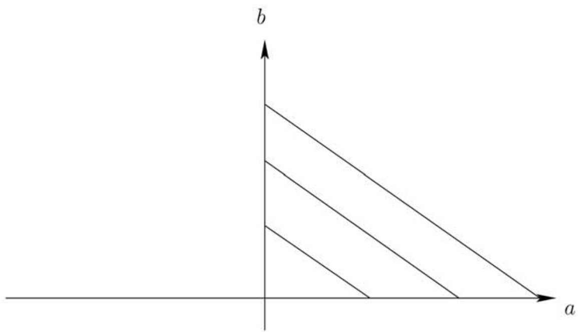

Figure 4. Stabilizing feedback vectors.

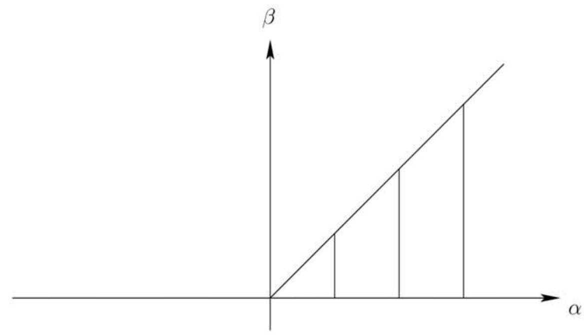

Figure 5. Stabilizing feedback vectors.

Consequently, the set of stabilizing feedback controllers for the original realization is given by the set

$$S_K = \left\{(a,b)\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}: a,b > 0\right\} = \{((a+b),b): a,b > 0\},$$

i.e., the set of stabilizing feedback vectors $K=(\alpha,\ \beta)$ is equal to the domain $\alpha > \beta,\ \beta > 0$. This domain is depicted in figure 5.

As we can see, in this case, the process of transforming back to the original realization narrows the cone of stabilizing feedback controllers.

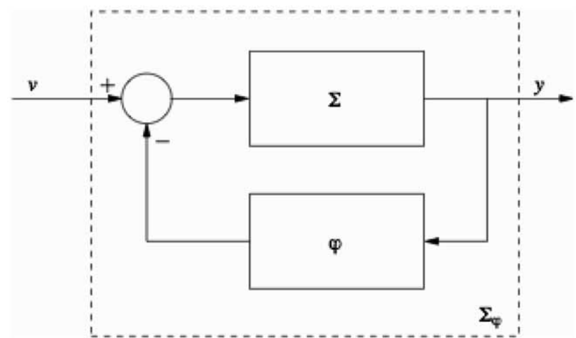

Figure 6. Output feedback.

4. Output feedback and tolerance horns

We turn now to an examination of the tolerance of high-gain linear dynamic output feedback compensators, using Theorem 1 as our starting point. We consider an output feedback configuration of figure 6. Here, the system Σ being controlled is a strictly causal time-invariant linear system, given by the equations

$$\Sigma: \quad \begin{aligned} \dot{x} &= Ax + Bu,\\ y &= cx. \end{aligned}$$

The feedback compensator φ is also a strictly causal linear time-invariant system, described by the equations

$$\varphi: \quad \begin{aligned} \dot{z} &= Dz + Ey,\\ w &= fz. \end{aligned}$$

The closed loop system is denoted by Σ_φ. It is convenient to describe the transfer matrices of Σ and of φ in terms of fraction representations over the polynomials $\Sigma = PQ^{-1}$, $\varphi = S^{-1}T$, where $\Sigma = PQ^{-1}$ is a right coprime fraction representation and $\varphi = S^{-1}T$ is a left coprime fraction representation. As usual, we assume that the transfer matrix of Σ is given, and we seek to compute the transfer matrix of φ. Our objective is to examine the tolerance allowed in the implementation of the feedback compensator φ.

Referring to figure 6, a direct calculation yields

$$\Sigma_\varphi = \Sigma[I + \varphi\Sigma]^{-1} = P[SQ + TP]^{-1}S.$$

Of course, we are interested only in internally stable control configurations, namely, configurations that can be implemented in a stable manner. (Needless to say, a transfer matrix is *stable* if all its poles are located in the open left half of the complex plane.) In this context, the following fact is well known (e.g., Hammer (1982)).

Proposition 1: *Figure 6 is internally stable if and only if the polynomial matrix $SQ + TP$ has a stable inverse matrix.*

4.1 *The case of single-input single-output systems*

To make the presentation more transparent, we examine first the case where Σ is a single-input single-output system. We show later that the general case of multi-input multi-output systems can be reduced to a collection of single-input single-output cases. Suppose then that the transfer function of the system being controlled is given in the form

$$\Sigma = \frac{p(s)}{q(s)},$$

where $p(s)$ and $q(s)$ are coprime polynomials. The coefficients of $p(s)$ and $q(s)$ are normalized so as to make the denominator $q(s)$ into a monic polynomial. In view of the fact that Σ is strictly causal, we have $\deg p(s) < \deg q(s)$. The objective is to find feedback compensators φ that internally stabilize Σ. Explicitly, since $\varphi(s)$ is strictly causal, we can always write

$$\varphi = \frac{\alpha(s)}{\beta(s)} = \frac{\alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n}{s^n + \beta_1 s^{n-1} + \cdots + \beta_n} \tag{35}$$

where $\alpha(s)$ and $\beta(s)$ are polynomials with $\beta(s)$ being a monic polynomial. In view of Proposition 1, in order to obtain an internally stable closed loop configuration, we need to find polynomials $\alpha(s)$ and $\beta(s)$ such that $\deg \alpha(s) < \deg \beta(s)$ and the polynomial

$$r(s) := \beta(s)q(s) + \alpha(s)p(s) \tag{36}$$

has all its roots in the open left half of the complex plane. Note that, in view of the strict causality of Σ and of φ, we have that $\deg\beta(s)q(s) > \deg\alpha(s)p(s)$; this implies that, in (36),

$$\deg r(s) = \deg\beta(s)q(s) = \deg\beta(s) + \deg q(s) \tag{37}$$

Consequently, $r(s)$ is a monic polynomial, since $\beta(s)$ and $q(s)$ are both monic polynomials. Note that $r(s)$ is the characteristic polynomial assigned to the closed loop system; it is the denominator of the closed loop system, and its roots are the poles of the closed loop system. The polynomial $r(s)$ is often specified as a design objective.

Now, let n be the dynamical order of the system Σ, so that $\deg q(s) = n$, and let $r(s)$ be the specified denominator of the closed loop system Σ_φ. The fact that the polynomials $p(s)$ and $q(s)$ are coprime implies that one can always find polynomials $\alpha'(s)$ and $\beta'(s)$ satisfying the equation

$$r(s) = \beta'(s)q(s) + \alpha'(s)p(s). \tag{38}$$

Using the polynomial division algorithm, we can find polynomials $a(s)$ and $\alpha(s)$ such that $\alpha'(s) = a(s)q(s) + \alpha(s)$, where

$$\deg\alpha(s) < n. \tag{39}$$

Then, (38) can be recast in the form $r(s) = [\beta'(s) + a(s)p(s)]q(s) + \alpha(s)p(s)$. Defining $\beta(s) := \beta'(s) + a(s)p(s)$, we have $r(s) = \beta(s)q(s) + \alpha(s)p(s)$. In order to be able to implement the compensator $\varphi(s)$ as a strictly causal system $\varphi(s) = \alpha(s)/\beta(s)$, we need

$$\deg\beta(s) > \deg\alpha(s). \tag{40}$$

Letting μ be the dynamical order of the compensator $\varphi(s)$, i.e., $\mu = \deg\beta(s)$, it follows from (39) that (40) is always valid when

$$\mu \geq n. \tag{41}$$

Now, let $\eta := \deg r(s)$ be the dynamical order of the closed loop system Σ_φ. Then, in view of (37), we have

$$\eta = \mu + n. \tag{42}$$

Combining (42) with (41), we obtain that $\eta - n \geq n$, or $\eta \geq 2n$; using (42), this yields

$$\mu \geq n.$$

Thus, by choosing the denominator $r(s)$ of the closed loop system to have the degree $\eta = 2n$, we can always derive a strictly causal feedback compensator φ of order n that stabilizes Σ. In other words, the following is true.

Proposition 2: *In figure 6, a strictly causal system Σ of dynamical order n can always be internally stabilized by a strictly causal feedback compensator φ of dynamical order n.*

In view of Proposition 2, we can confine our attention to feedback compensators of dynamical order n when considering the stabilization of a system of order n. In such case, the dynamical order of the closed loop system Σ_φ will be $2n$. Consider then a strictly causal feedback compensator $\varphi = \alpha(s)/\beta(s)$ of order n. Recalling that the denominator $\beta(s)$ was normalized as a monic polynomial and that the degree of $\alpha(s)$ is strictly lower than the degree of $\beta(s)$, we write

$$\alpha(s) = \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n,$$
$$\beta(s) = s^n + \beta_1 s^{n-1} + \cdots + \beta_n.$$

Then, we can represent the compensator φ by a real column vector $v(\varphi)$ of dimension $2n$, where the top n entries are given by the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the numerator polynomial, and the bottom n entries are given by the coefficients $\beta_1, \beta_2, \ldots, \beta_n$ of the denominator polynomial, i.e.,

$$v(\varphi) = (\alpha_1, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n)^T. \tag{43}$$

Further, we have seen that under these circumstance the denominator of the closed loop system Σ_φ is the monic polynomial $r(s)$ of degree $2n$, i.e., $r(s) = s^{2n} + r_1 s^{2n-1} + r_2 s^{2n-2} + \cdots + r_{2n}$. This polynomial can be represented by the column vector

$$v(r) = (r_1, r_2, \ldots, r_{2n})^T.$$

Writing $q(s) = s^n + q_1 s^{n-1} + \cdots + q_n$ and $p(s) = p_1 s^{n-1} + p_2 s^{n-2} + \cdots + p_n$ for the denominator and the numerator polynomials, respectively, of the given system Σ, it follows from (36) that $s^{2n} + r_1 s^{2n-1} + r_2 s^{2n-2} + \cdots + r_{2n} = (s^n + \beta_1 s^{n-1} + \cdots + \beta_n) \times (s^n + q_1 s^{n-1} + \cdots + q_n) + (\alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n) \times (p_1 s^{n-1} + p_2 s^{n-2} + \cdots + p_n)$. Equating the coefficients of corresponding powers, this leads to the relations

$$\left.\begin{aligned} r_1 &= q_1 + \beta_1, \\ r_2 &= q_2 + \beta_1 q_1 + \beta_2 + \alpha_1 p_1, \\ r_3 &= q_3 + \beta_1 q_2 + \beta_2 q_1 + \beta_3 + \alpha_1 p_2 + \alpha_2 p_1, \\ &\ldots \\ r_n &= q_n + \beta_1 q_{n-1} + \ldots + \beta_{n-1} q_1 + \beta_n \\ &\quad + \alpha_1 p_{n-1} + \ldots + \alpha_{n-1} p_1, \\ r_{n+1} &= \beta_1 q_n + \cdots + \beta_n q_1 + \alpha_1 p_n + \cdots + \alpha_n p_1, \\ &\ldots \\ r_{2n} &= \beta_n q_n + \alpha_n p_n. \end{aligned}\right\} \quad (44)$$

The relations (44) can be rewritten in the form

$$v(r) = Dv(\varphi) + \vartheta, \quad (45)$$

where $\vartheta = (q_1, \ldots, q_n, 0, 0, \ldots, 0)^T$ and D is the $2n \times 2n$ matrix

$$D = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ p_1 & 0 & \cdots & 0 & 0 & q_1 & 1 & 0 & \cdots & 0 & 0 \\ p_2 & p_1 & 0 & \cdots & 0 & q_2 & q_1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n-1} & \cdots & p_2 & p_1 & 0 & q_{n-1} & \cdots & q_3 & q_2 & q_1 & 1 \\ p_n & p_{n-1} & \cdots & p_2 & p_1 & q_n & q_{n-1} & \cdots & q_3 & q_2 & q_1 \\ 0 & 0 & 0 & \cdots & 0 & p_n & 0 & 0 & \cdots & 0 & q_n \end{pmatrix}. \quad (46)$$

As we can see, the matrix D is determined by the given coefficients $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$ of the transfer function of Σ. We refer to D as the *transformation matrix* of Σ. Defining the vector

$$\rho := v(r) - \vartheta, \quad (47)$$

we obtain the linear relationship

$$\rho := Dv(\varphi). \quad (48)$$

An important feature of this relationship is the following fact.

Proposition 3: *The matrix D of* (46) *is non-singular when* Σ *is not the zero system.*

Proof: In view of the relation (48), our proof will conclude upon showing that $\rho = 0$ if and only if $v(\varphi) = 0$. The 'if' direction follows immediately from (48); thus, it only remains to prove that $\rho = 0$ implies that $v(\varphi) = 0$. We prove the latter by examining the polynomial equation

$$\beta(s)q(s) + \alpha(s)p(s) = r(s). \quad (49)$$

Assume then that $\rho = 0$. Recalling that $\beta(s)$ is a monic polynomial, write $\beta(s) = s^n + \beta'(s)$ so that

$$\deg\beta'(s) \leq n - 1$$

Substituting into (49), we get

$$s^n q(s) + \beta'(s)q(s) + \alpha(s)p(s) = r(s). \quad (50)$$

Defining the polynomial $\rho(s) := \rho_1 s^{2n-1} + \cdots + \rho_{2n}$, it follows from (47) that $\rho(s) = r(s) - s^n q(s)$. Using (50), we obtain $\rho(s) = \beta'(s)q(s) + \alpha(s)p(s)$. When $\rho = 0$, we have $\rho(s) = 0$, so that $\beta'(s)q(s) + \alpha(s)p(s) = 0$, or

$$\beta'(s)q(s) = -\alpha(s)p(s). \quad (51)$$

The last equation describes a common multiple of $q(s)$ and $p(s)$. Now, let $\mu(s)$ be the least common multiple of $p(s)$ and $q(s)$. Then, since $p(s)$ and $q(s)$ are coprime polynomials, $\mu(s)$ is an associate of the product $p(s)q(s)$. Hence,

$$\deg\mu(s) = \deg p(s) + \deg q(s) = \deg p(s) + n. \quad (52)$$

Further, since $\deg\beta(s) = n$, the strict causality of φ implies that $\deg\alpha(s) < n$. Consequently, $\deg\alpha(s)p(s) < \deg p(s) + n$, which shows that $\alpha(s)p(s)$ cannot be a non-zero multiple of $\mu(s)$. As a result, the validity of both (51) and (52) implies that

$$\beta'(s)q(s) = -\alpha(s)p(s) = 0.$$

In view of the fact that Σ is not the zero system, we have that $p(s) \neq 0$ and $q(s) \neq 0$, and it follows that $\alpha(s) = 0$ and $\beta'(s) = 0$. But then, $v(\varphi) = 0$, and our proof concludes. □

In view of (45) and Proposition 3, we can express the coefficients $v(\varphi)$ of the feedback controller φ in terms of the desired denominator coefficients $v(r)$ of the closed loop system in the form

$$v(\varphi) = D^{-1}[v(r) - \vartheta]. \quad (53)$$

Example 8: Consider an order 2 system with the transfer function

$$\Sigma = \frac{3s + 2}{s^2 + s + 1},$$

and let $r(s) = s^4 + r_1 s^3 + r_2 s^2 + r_3 s + r_4$ be the desired characteristic polynomial of the closed loop system Σ_φ (here, r_1, r_2, r_3 and r_4 are given real numbers). To find a strictly causal feedback compensator φ that implements this characteristic polynomial for the closed loop system, write

$$\varphi(s) = \frac{\alpha_1 s + \alpha_2}{s^2 + \beta_1 s + \beta_2}. \quad (54)$$

Then, $v(\varphi) = (\alpha_1, \alpha_2, \beta_1, \beta_2)^T$. Using (36), we get $r(s) = \alpha(s)p(s) + \beta(s)q(s) = (\alpha_1 s + \alpha_2)(3s + 2) + (s^2 + \beta_1 s + \beta_2) \times (s^2 + s + 1) = s^4 + (\beta_1 + 1)s^3 + (3\alpha_1 + \beta_1 + \beta_2 + 1)s^2 + (2\alpha_1 + 3\alpha_2 + \beta_1 + \beta_2)s + (2\alpha_2 + \beta_2)$.

Equating coefficients of corresponding powers of s, we get

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ 3\alpha_1 + \beta_1 + \beta_2 \\ 2\alpha_1 + 3\alpha_2 + \beta_1 + \beta_2 \\ 2\alpha_2 + \beta_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Comparing to (45), we obtain

$$v(r) = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \text{ and } \vartheta = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

A direct calculation verifies that D is indeed an invertible matrix. Thus, we can solve for the coefficients α_1, α_2, β_1, β_2 of the transfer function φ for any choice of r_1, r_2, r_3 and r_4. This implies that a compensator φ of the form (54) can assign any monic polynomial of degree 4 as the denominator of the closed loop system Σ_φ.

Let us turn now to the issue of stabilizing the given system Σ by using figure 6. To accomplish this objective, it is sufficient to assign as the characteristic polynomial of the closed loop system a monic polynomial with coefficients in the set $V(\delta, M)$ of Lemma 3. In connection with (47), define the set

$$\psi(\delta, M) := V(\delta, M) - \vartheta, \tag{55}$$

which is obtained by simply shifting $V(\delta, M)$ by ϑ; recall that ϑ is given in terms of the parameters of the system Σ. Clearly, the process of shifting $V(\delta, M)$ by ϑ has no impact on its inner span. Consequently, the set $\psi(\delta, M)$ has the same inner span as the set $V(\delta, M)$.

Further, considering (53), we conclude that any feedback compensator φ of the form (35) with coefficients vector

$$v(\varphi) \in D^{-1}[\psi(\delta, M)] \tag{56}$$

creates an internally stable closed loop system Σ_φ. In close analogy to our derivation of Theorem 2, we obtain then the following characterization of the tolerance permitted in the implementation of a stabilizing output feedback compensator φ.

Proposition 4: *Let Σ be a strictly causal single-input single-output system, and let D be the transformation matrix of Σ. Let $\mathcal{F}_n$ be the set of all strictly causal feedback controllers φ of dynamical order n that internally stabilize the system Σ, and let V_n be the set of all $2n$ dimensional vectors $v(\varphi)$, $\varphi \in \mathcal{F}_n$. Then, V_n includes a virtual horn whose span is the column span of the matrix D^{-1}.*

The fact that a polynomial whose roots are inside the open left half of the complex plane must have strictly positive coefficients directly implies the following.

Corollary 3: *Let Σ be a strictly causal scalar system with the transformation matrix D. Let $\mathcal{F}_n$ be the set of all strictly causal feedback controllers φ of dynamical order n that internally stabilize Σ, and let V_n be the set of all $2n$ dimensional vectors $v(\varphi)$, $\varphi \in \mathcal{F}_n$. Then, the horn of V_n is virtually equal to the horn $D^{-1}R^{+n}$.*

Corollary 3 shows that the transformation matrix D of the given system Σ characterizes the accuracy required of an internally stabilizing output feedback controller for Σ. Our next objective is to generalize these statements to the case of multi-input multi-output systems.

4.2 *Multivariable systems*

Consider a strictly causal system Σ with m input variables and p output variables being controlled by a strictly causal output feedback controller φ in figure 6. The controller φ is then represented by an $m \times p$ transfer matrix $\varphi(s)$, whose entries $\varphi_{ij}(s)$ are strictly causal rational functions. With each non-zero entry $\varphi_{ij}(s)$, we associate a vector $v(\varphi_{ij})$ of real numbers, following the process used in (43). It is convenient to combine all these vectors into one long vector $v(\varphi)$. The vector $v(\varphi)$ contains then all the design parameters required for the stabilization of the system Σ. We are interested in the tolerances around the values of these parameters. The next statement generalizes Proposition 4 to the case of multi-input multi-output systems.

Theorem 3: *Let Σ be a strictly causal system of order n, let $\mathcal{F}_n$ be the set of all strictly causal feedback controllers φ of dynamical order n that internally stabilize Σ, and let V be the set of all vectors $v(\varphi)$, $\varphi \in \mathcal{F}_n$. Then, V includes a virtual horn H of strictly positive span.*

The calculation of the span of the horn H mentioned in the theorem is described in the proof (see (67) below).

Proof of Theorem 3: Consider a strictly causal system with a $p \times m$ transfer matrix Σ, and let $\Sigma = PQ^{-1}$ be a right coprime fraction representation over the polynomials. Here, P is a $p \times m$ polynomial matrix, Q is an $m \times m$ polynomial matrix that is invertible over the rational functions, and P and Q are right coprime. For notational simplicity, we assume here that $p \geq m$; the case $p < m$ is treated similarly. Consider the control of Σ by a dynamic output feedback compensator φ as depicted in figure 6, so that φ has an $m \times p$ transfer matrix. It is convenient to represent φ in terms of a left coprime fraction representation over the polynomials $\varphi = S^{-1}T$. In view of Proposition 1, internal stability of figure 6 is achieved when the determinant of the polynomial matrix $M := TP + SQ$ has all its roots inside the open left half of the complex plane. In the course of the proof, we show that the case of a multivariable system can be reduced to several single-input single-output systems. This will allow us to generalize the results of § 4.1 to multivariable systems.

To this end, let $\Omega_s^- R$ be the set of all scalar, stable, and causal rational functions; recall that $\Omega_s^- R$ forms a principal ideal domain (e.g., Hammer (1983a)). An *invertible* in $\Omega_s^- R$ is a non-zero element for which the inverse is also stable and causal. A *unimodular* matrix A over $\Omega_s^- R$ is an invertible matrix of rational functions for which all entries (of A and) of A^{-1} belong to $\Omega_s^- R$. A matrix B of rational functions is *bicausal* if it is invertible, and if all entries of B and of B^{-1} are causal. Similarly, a matrix C of rational functions is *bistable* if all entries of C and of C^{-1} have all their poles in the open left half of the complex plane. Clearly, a unimodular matrix over $\Omega_s^- R$ is bicausal and bistable.

From the fact that the given transfer matrix Σ is rational, it follows that there is a bicausal element $\psi(s) \in \Omega_s^- R$ for which the product $f(s) := \Sigma\psi(s)$ is a stable and causal transfer matrix. Then, $f(s)$ is a matrix over the principal ideal domain $\Omega_s^- R$. By the Hermite Normal Form Theorem (e.g., Macduffee (1946)), it follows that there is a unimodular matrix ℓ over $\Omega_s^- R$ for which the product $\ell f(s)$ is in lower triangular form (i.e., is zero below its main diagonal). Since $\psi(s)$ is a scalar, the transfer matrix

$$\Sigma' := \ell\Sigma = \left(\frac{1}{\psi(s)}\right)\ell f(s) \tag{57}$$

is still in lower triangular form. Furthermore, since ℓ is bicausal and Σ is strictly causal, it follows that Σ' is strictly causal as well.

Next, let $\Sigma' = P'Q'^{-1}$ be a right coprime fraction representation over the polynomials. Applying again the Hermite Normal Form Theorem, we can derive a polynomial unimodular matrix N for which the product $Q'' := Q'N$ is in lower triangular form. Define the matrix $P'' := P'N$, so that

$$\Sigma' = P''Q''^{-1}. \tag{58}$$

Noting that Σ' and Q'' are both lower triangular matrices and that $P'' = \Sigma'Q''$, we conclude that the matrix P'' is also in lower triangular form. Thus, P'' and Q'' are both polynomial matrices in lower triangular form.

To simplify our discussion while sacrificing generality only slightly, we make the following assumption, which is valid in the generic case of polynomial matrices P'' and Q'' (we indicate later how to handle cases in which this assumption is not valid). Let p_{ij} and q_{ij} be the i, j entries of the matrices P'' and Q'', respectively.

Assumption 1 (generic case): *Corresponding main diagonal entries of P'' and Q'' are coprime polynomials, i.e., p_{ii} and q_{ii} are coprime for all $i = 1, 2, \ldots, m$.*

Now, since Q'' is lower triangular, so is its inverse matrix Q''^{-1}, and the ith entry on the main diagonal of Q''^{-1} is $1/q_{ii}$, $i = 1, 2, \ldots, m$. Thus, recalling that P'' is lower triangular as well, it follows that the main diagonal entries of the product $\Sigma' = P''Q''^{-1}$ are p_{ii}/q_{ii}, $i = 1, 2, \ldots, m$. In view of the strict causality of Σ', this leads us to the conclusion that $\deg p_{ii} < \deg q_{ii}$, $i = 1, \ldots, m$. Thus, the system $f_{ii} := p_{ii}/q_{ii}$ satisfies the requirements of Proposition 2. Consequently, there is a strictly causal output feedback compensator φ'_{ii} that internally stabilizes the scalar system represented by the transfer function f_{ii}. Let $\varphi'_{ii} = \alpha_{ii}/\beta_{ii}$ be a coprime polynomial fraction representation, where we take $\beta_{kk} := 1$ if $\varphi'_{kk} = 0$. The fact that φ'_{ii} internally stabilizes the scalar system f_{ii} implies that the polynomial

$$\mu_{ii} := \alpha_{ii}p_{ii} + \beta_{ii}q_{ii} \tag{59}$$

has all its roots in the open left half of the complex plane, $i = 1, \ldots, m$.

Next, build the $m \times m$ diagonal matrix B, having $\beta_{11}, \beta_{22}, \ldots, \beta_{mm}$ on its diagonal, and note that B is a non-singular matrix. Build the $m \times p$ diagonal matrix A, having the entries $\alpha_{11}, \alpha_{22}, \ldots, \alpha_{mm}$ on its main diagonal and zeros everywhere else (recall that we consider the case $p \geq m$). Taking into account the fact that P'' and Q'' are triangular matrices, that A and B are diagonal matrices, and that (59) is valid, it follows that the matrix

$$M := AP'' + BQ'' \tag{60}$$

is an $m \times m$ triangular matrix with the entries $\mu_{11}, \mu_{22}, \ldots, \mu_{mm}$ on its main diagonal. Consequently, $\det M = \mu_{11}, \mu_{22}, \ldots, \mu_{mm}$, namely, $\det M$ is a polynomial having all its roots in the open left half of the complex plane. By Proposition 1, we conclude then that the $m \times p$ diagonal feedback compensator

$$\varphi' := B^{-1}A = \begin{pmatrix} \varphi'_{11} & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \varphi'_{22} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \varphi'_{mm} & 0 & \cdots & 0 \end{pmatrix}$$

internally stabilizes the system Σ'. Applying now Proposition 4 and Corollary 3 to each one of the scalar feedback compensators φ'_{ii}, $i = 1, 2, \ldots, n$, it follows that the set of permissible coefficients of each non-zero entry of φ' virtually includes a horn with strictly positive span.

Returning now to the given system Σ, we have from (57) and (58) that $\Sigma = (\ell^{-1}P'')Q''^{-1}$. Defining the matrices

$$P := \ell^{-1}P'' \text{ and } Q := Q'',$$

it follows that $P'' = \ell P$ and (60) takes the form

$$(A\ell)P + BQ = M. \tag{61}$$

Now, let $\Omega_s R$ be the ring of all stable scalar transfer functions. Recalling that ℓ is bistable and that M is a polynomial matrix whose determinant has only stable roots, it follows that the matrices ℓ and M are both unimodular over the ring $\Omega_s R$. Furthermore, the fact that ℓ is unimodular over $\Omega_s R$, together with the fact that P'' and Q'' are coprime polynomial matrices, implies that P and Q are right coprime matrices over $\Omega_s R$. Thus, (61) represents a solution of the coprimeness equation (the Bezout Identity) over the ring $\Omega_s R$. In view of Hammer (1983b), this implies that the feedback compensator φ given by

$$\varphi := B^{-1}(A\ell) = (B^{-1}A)\ell = \varphi'\ell$$

internally stabilizes the original system Σ.

Note that, since ℓ is bicausal and $B^{-1}A$ is strictly causal, it follows that φ is strictly causal as well. Furthermore, since $B^{-1}A$ is diagonal, each entry of φ is obtained by multiplying one entry of the diagonal compensator φ' by one of the entries of ℓ. Specifically, letting ℓ_{ij} be the i, j entry of ℓ, it follows that the i, j entry of φ is given by

$$\varphi_{ij} = \varphi'_{ii}\ell_{ij}, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, p.$$

Let us write the terms as coprime polynomial fraction representations in the form

$$\varphi'_{ii} = \frac{\alpha_{ii}}{\beta_{ii}}, \quad \ell_{ij} = \frac{\lambda_{ij}}{\delta_{ij}}, \tag{62}$$

where α_{ii}, β_{ii} and λ_{ij}, δ_{ij} are pairs of coprime polynomials. Defining the two polynomials $\rho_{ij} := \alpha_{ii}\lambda_{ij}$ and $\eta_{ij} = \beta_{ii}\delta_{ij}$, we obtain that $\varphi_{ij} = \rho_{ij}/\eta_{ij}$. In analogy with (43), we create a real vector $v(\varphi_{ij})$ from the coefficients of the two polynomials ρ_{ij} and η_{ij}. Similarly, we create a real vector $v(\varphi'_{ii})$ from the coefficients of the polynomials α_{ii} and β_{ii}.

Fact 1: $v(\varphi_{ij})$ is a linear function of $v(\varphi'_{ii})$, as long as the dynamical orders of φ_{ij} and φ'_{ii} remain constant.

Proof of Fact 1: Consider two rational functions

$$\varphi'_{1,ii} = \frac{\alpha_{1,ii}}{\beta_{1,ii}} \text{ and } \varphi'_{2,ii} = \frac{\alpha_{2,ii}}{\beta_{2,ii}}$$

of the same dynamical order, and let a and b be two real numbers. Also, denote

$$\varphi_{1,ij} := \varphi'_{1,ii}\ell_{ij}, \ \varphi_{2,ij} := \varphi'_{2,ii}\ell_{ij} \tag{63}$$

and assume that $\varphi_{1,ij}$ and $\varphi_{2,ij}$ have the same dynamical order. Explicitly, by (62), we can write

$$\varphi_{1,ij} = \frac{\lambda_{ij}\alpha_{1,ii}}{\delta_{ij}\beta_{1,ii}} \text{ and } \varphi_{2,ij} = \frac{\lambda_{ij}\alpha_{2,ii}}{\delta_{ij}\beta_{2,ii}}. \tag{64}$$

Now, according to the construction (43) of the real vector $v(\varphi)$, it follows that the rational function φ'_{ii} that corresponds to the combination $av(\varphi'_{1,ii}) + bv(\varphi'_{2,ii})$ is given by $\varphi'_{ii} := (a\alpha_{1,ii} + b\alpha_{2,ii})/(a\beta_{1,ii} + b\beta_{2,ii})$. Using (62), this leads to

$$\varphi_{ij} = \ell_{ij}\varphi'_{ii} = \frac{\lambda_{ij}(a\alpha_{1,ii} + b\alpha_{2,ii})}{\delta_{ij}(a\beta_{1,ii} + b\beta_{2,ii})} = \frac{a(\lambda_{ij}\alpha_{1,ii}) + b(\lambda_{ij}\alpha_{2,ii})}{a(\delta_{ij}\beta_{1,ii}) + b(\delta_{ij}\beta_{2,ii})}.$$

Using again the construction (43) and (64), it follows from the last equality that $v(\varphi_{ij}) = av(\varphi_{1,ij}) + bv(\varphi_{2,ij})$. This shows that our fact is valid. □

In view of Fact 1, under conditions of constant dynamical order, there is a matrix $L(i, j)$ such that

$$v(\varphi_{ij}) = L(i,j)v(\varphi'_{ii}), \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, p. \tag{65}$$

Furthermore, the following is true.

Fact 2: If $\ell_{ij} \neq 0$, then the matrix $L(i, j)$ of (65) is non-singular.

Proof: Assume that there are two vectors $v(\varphi'_{1,ii})$ and $v(\varphi'_{2,ii})$ for which $L(i,j)v(\varphi'_{1,ii}) = L(i,j)v(\varphi'_{2,ii}) =: v(\varphi_{ij})$. Using (63), this yields $\varphi_{ij} = \varphi'_{1,ii}\ell_{ij} = \varphi'_{2,ii}\ell_{ij}$; since $\ell_{ij} \neq 0$, we conclude that $\varphi'_{1,ii} = \varphi'_{2,ii}$. Thus, we must have $v(\varphi'_{1,ii}) = v(\varphi'_{2,ii})$, and it follows that $L(i, j)$ is a non-singular matrix. □

Next, using (59) and implementing the process outlined in (53), (55), (56), and Proposition 4, we derive the set $\psi_{ii}(\delta, M)$ having the following feature: all dynamic feedback compensators φ'_{ii} whose coefficients satisfy

$$v(\varphi'_{ii}) \in D_{ii}^{-1}[\psi_{ii}(\delta, M)],$$

internally stabilize the scalar system $f_{ii} = p_{ii}/q_{ii}$. Combining this with (65), we conclude that any feedback compensator φ whose i, j entry has coefficients $v(\varphi_{ij})$ satisfying

$$v(\varphi_{ij}) \in (L(i,j)D_{ii}^{-1})[\psi_{ii}(\delta, M)] \tag{66}$$

internally stabilizes the given system Σ. Since the matrix $L(i, j)$ is not singular by Fact 2, this shows that the coefficients of each non-zero entry of the internally stabilizing feedback compensator φ include a virtual horn of strictly positive span.

More explicitly, let ρ_{ij} be the inner span of the cone $(L(i,j)D_{ii}^{-1})[\psi_{ii}(\delta, M)]$. As the matrix $L(i, j)$ is non-singular for all non-zero entries of ℓ_{ij} (Fact 2), we conclude that $\rho_{ij} > 0$ for all nonzero entries of φ. Defining

$$\rho := \min\{\rho_{ij} : \ell_{ij} \neq 0,\ i = 1, \ldots, m,\ j = 1, \ldots, p\}, \quad (67)$$

we have that $\rho > 0$. Thus, the nonzero entries of the feedback compensator φ allow a tolerance that corresponds to a virtual horn with the inner span ρ. This concludes the proof of Theorem 3 in the generic case (i.e., under Assumption 1).

Finally, we provide a sketch of the proof of Theorem 3 in the non-generic case, i.e., when Assumption 1 is not valid. Recall the bicausal element $\psi(s) \in \Omega_s^- R$ for which the transfer matrix $f(s) := \psi(s)\Sigma$ is stable. As $\Omega_s^- R$ is a principal ideal domain, e.g., Hammer (1983a), there are unimodular matrices ℓ_1 and ℓ_2 over $\Omega_s^- R$ for which the transfer matrix $\ell_1 f(s)\ell_2$ is in Smith normal form over $\Omega_s^- R$ (Macduffee 1946). Dividing the last expression by the scalar $\psi(s)$, we obtain the diagonal transfer matrix

$$\Sigma'' := \ell_1 \Sigma \ell_2. \quad (68)$$

Let $\sigma_{11}, \sigma_{22}, \ldots, \sigma_{mm}$ be the diagonal entries of Σ''; note that $\sigma_{11}, \sigma_{22}, \ldots, \sigma_{mm}$ are all strictly causal, since Σ is strictly causal and ℓ_1 and ℓ_2 are both bicausal.

Next, let $\Sigma'' = P''Q''^{-1}$ be a diagonal coprime fraction representation over the polynomials (i.e., P'' and Q'' are diagonal polynomial matrices). The corresponding diagonal entries of P'' and of Q'' are then coprime polynomials. Let E and F be diagonal polynomial matrices for which the combination

$$EP'' + FQ'' = M \quad (69)$$

is a diagonal polynomial matrix whose determinant has all its roots in the open left half of the complex plane. In view of Proposition 2, we can choose the diagonal matrices E and F so that the transfer matrix $\phi'' := F^{-1}E$ is strictly causal. Further, using (68), we can write $\Sigma = \ell_1^{-1}\Sigma''\ell_2^{-1} = (\ell_1^{-1}P'')(\ell_2 Q'')^{-1} = PQ^{-1}$, where

$$P := \ell_1^{-1}P'' \text{ and } Q := \ell_2 Q''.$$

Referring now to (69), we can write $M = (E\ell_1)P + (F\ell_2^{-1})Q$, from which we conclude, as earlier in this proof, that

$$\phi := (F(\ell_2)^{-1})^{-1}E\ell_1 = \ell_2\phi''\ell_1$$

is a strictly causal feedback compensator that internally stabilizes the system Σ. Finally, since the non-zero coefficients of ϕ'' include the horn of a cone with strictly positive span, an argument based on the one leading to (66) implies that the same also holds true for the nonzero coefficients of ϕ. This concludes our proof. □

5. Conclusion

We have discussed the accuracy required of stabilizing feedback controllers for linear time-invariant systems. We have seen that each system is naturally associated with a tolerance cone, which describes the class of high-gain feedback controllers that stabilize the system. The tolerance cone can be computed from the given description of the system being controlled, and its vertex angle determines the relative (or "percentage") accuracy required of the parameters of a stabilizing high-gain feedback controller. In the case of static state feedback, the vertex angle of the tolerance cone is small when the normalized controllability matrix of the controlled system is close to being singular. This indicates that higher accuracy is required of the feedback parameters when controlling a system that is close to losing reachability, making it harder to stabilize input/state systems that are nearly non-reachable. The situation for dynamic output feedback is analogous.

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