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Handling disturbances in nonlinear control: the use of state feedback

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The problem of controlling a nonlinear system in the presence of disturbances and modelling inaccuracies is considered. The objective is to design a static state feedback controller that controls the system so that the combined effect of all disturbances and inaccuracies on the response of the closed loop system is below a specified bound. The results include a characterization of the largest disturbance amplitude for which the design objective can be met. This result can be used to find the largest discretization step for a digital controller. Note that a larger discretization step lowers the computational burden of the controller. By exploring a basic connection to reachability, a procedure for the computation of appropriate static state feedback controllers is described. The state feedback controllers are calculated through the solution of a system of algebraic inequalities.

1. Introduction

A common difficulty in the utilization of digital controllers for nonlinear systems is the extensive computational burden imposed by the controller. Even with modern computer systems, the implementation of controllers for nonlinear systems of moderate to high dimensionality is a daunting task. This difficulty has only been partly alleviated by modern design techniques, and it still constitutes an obstacle in the practice of nonlinear control engineering.

An important parameter in the design of a discrete controller for a system with continuously valued signals is the size of the discretization step used in the analogue-to-digital conversion process. For a fixed signal amplitude, a larger discretization step leads to fewer points in the discrete space over which the discrete controller operates, thus lowering the computational requirements of the controller. The present paper concentrates on the problem of finding the largest discretization step that is compatible with specified performance requirements. At the same time, it also describes the design of a controller appropriate for this discretization step.

The system being controlled is a nonlinear discrete-time system Σ, represented by a recursion of the form

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) \\ y_k &= x_k, \quad k = 0, 1, 2, \ldots \end{aligned} \tag{1}$$

Here, the initial condition is x_0. The variable x_k is an n-dimensional real vector, usually called the 'state' of Σ at the step k; the input value of Σ at the step k is given by u_k, an m-dimensional real vector. The function f is called the *recursion function* of Σ, and is required to be a continuous function. We concentrate here on the control of systems whose state is provided as output, so the output value y_k at the step k is equal to the state of the system at that step.

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The system Σ is subject to a variety of disturbances and modelling inaccuracies. These consist of an input disturbance signal v_1, an output disturbance signal v_3, and a modelling error v_2. When these disturbances and errors are taken into account, the recursive representation of the system becomes

$$\begin{aligned} x_{k+1} &= f(x_k, u_k + v_{1k}) + v_{2k} \\ y_k &= x_k + v_{3k}, \quad k = 0, 1, 2, \ldots \end{aligned} \tag{2}$$

Here, v_{1k} is the value of the disturbance signal v_i at the step k. The only *a priori* information available about the disturbances v_1, v_2, and v_3 is a bound on their largest amplitude. Thus, we assume that there is a specified real number $d_0 > 0$ such that the amplitudes of the disturbances v_1, v_2, and v_3 do not exceed d_0. No other assumptions are made about the nature of the disturbances.

The class of controllers considered in the present paper is the class of static state feedback controllers, as represented by figure 1.

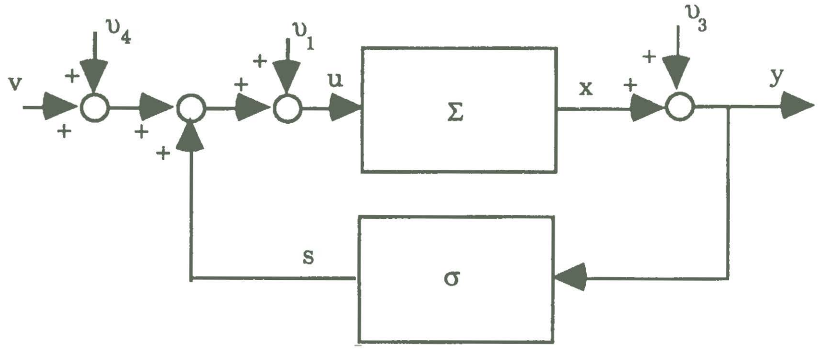

Figure 1.

Here, σ is a static state feedback function, v is an external reference input, and v_4 is an input disturbance. The amplitude of the disturbance v_4 is also bounded by $d_0 > 0$, as the other disturbance amplitudes are. The closed loop system described by the diagram is denoted by Σ_σ.

We emphasize that the disturbance signals v_1, v_2, v_3, and v_4 are not assumed to be infinitesimal. The design framework development below is of a global nature, and allows the treatment of large disturbances and deviations.

Generally speaking, the disturbances v_1, v_2, v_3, and v_4 may originate from a variety of different sources. When the feedback function σ is implemented on a digital computer, one important disturbance source is the discretization noise, caused by the analogue-to-digital and digital-to-analogue conversion processes. The maximal amplitude of this disturbance source is the size d_d of the discretization step. When this disturbance is added to the existing disturbances in the system, one obtains the bound

$$d := d_0 + d_d \tag{3}$$

on the amplitudes of the combined disturbances. For the sake of simplicity, we have assumed here that the same discretization step size is used for all signals. Similar techniques apply to the more general case where each signal has its own discretization step size.

The objective is to find the largest possible value of d that is compatible with given design specifications. Once the maximal value of d has been found, one can obtain the size of the largest permissible discretization step from the equality

$$d_{\mathrm{d}} = d - d_0 \tag{4}$$

Note that a larger discretization step reduces the computational burden required of the digital controller that implements the feedback function σ.

The specific design problem considered here is the calculation of a state feedback function σ that drives the system Σ along a prescribed nominal path. The nominal path is obtained when the external reference signal v of the closed loop system is set to zero, and no disturbances are active.

When active, the disturbances υ_1, υ_2, υ_3, and υ_4 may cause the closed loop system Σ_σ to deviate from its specified nominal path. The largest magnitude of this deviation determines the performance accuracy of the closed loop system. The design requirement is that the deviation does not exceed a prescribed bound $\Delta > 0$. Our aim is to find the largest disturbance amplitude bound d for which there is a feedback function σ that fulfils this requirement. When the maximal value of d is substituted into (4), one obtains the size of the largest permissible discretization step d_{d} which is compatible with the design requirements.

After possibly inducing a shift on the state variables of Σ, we shall assume that the required nominal path is the zero output sequence. In these terms, we can state our design objective as follows.

Design objective:

(a) *Given a real number $\Delta > 0$, find the largest real number $d > 0$ for which there is a state feedback function σ satisfying the property:*

(*) *The output amplitude of the closed loop system Σ_σ does not exceed Δ for any disturbance signals υ_1, υ_2, υ_3, and υ_4 of amplitude not exceeding d (if such a d exists)*; *and*

(b) *Construct a state feedback function σ with the property* (*).

The techniques developed in the paper are general in nature, and can be utilized for the design of controllers that fulfil other design objectives as well. However, we shall concentrate solely on the design objective listed above. Throughout our discussion, we assume that the external reference signal v is set to its zero nominal value; possible deviations from this value are represented by the disturbance signal υ_4.

Our discussion provides a link between the accuracy requirements imposed on the performance of the closed loop system, and the computational burden required for the implementation of the controller σ. A larger value of the closed loop error Δ normally leads to a larger value of the disturbance amplitude bound d, and whence, via (4), to a larger discretization step d_{d}. This reduces the number of points in the discrete implementation of σ, and lowers the computational burden.

The bound d on the maximal permissible disturbance amplitude is characterized in section 3. In principle, the calculation of the bound d involves the solution of a set of algebraic inequalities that are derived from the given recursion function f of the system Σ. Section 3 also contains the construction of state feedback functions σ that permit disturbances of amplitudes up to the maximal value d.

As mentioned earlier, our main interest here is in the derivation of state feedback functions for digital computer implementation, as other implementations of nonlinear controllers are usually impractical. In such an implementation, the feedback function σ operates over a discrete grid, where the interval size of the grid is given by the discretization interval d_{d}. Thus, there is no need to require the function σ to be a continuous function, and we can permit σ to possess jumps. The only requirement imposed on the feedback function σ is that it control the system Σ so that the output deviations caused by permissible disturbances do not exceed the specified bound Δ. Allowing the function σ to be discontinuous considerably simplifies its construction, while being of no adverse consequence in a discrete implementation. (The derivation of continuous feedback functions is discussed in Hammer (1989b).)

In general terms, the discrete implementation of the feedback function σ is constructed as follows. First, one calculates the permissible disturbance amplitude d, and then, through (4), the permissible discretization step d_{d}. This defines the step size of the grid over which the values of σ need to be calculated. The values of σ over the grid are then calculated on a point by point basis, from the solution of a set of algebraic inequalities (see sections 3 and 4 below). Since most practical systems operate over bounded spaces, the values of σ need to be calculated only over a finite set of argument values; the number of such argument values is clearly smaller for a larger discretization step size d_{d}.

In section 4 we show that the calculation of permissible disturbance amplitudes, as well as the derivation of appropriate state feedback functions σ, is simplest when the system Σ satisfies certain reachability requirements. This observation generalizes to the nonlinear case a well known principle of linear control theory, where the design of state feedback is closely linked to reachability properties.

The notion of nonlinear reachability used in our discussion is based directly on properties of the recursion function f of the system. Specifically, let

$$f^{\,n}(x_0, u_0, \ldots, u_{n-1}) := f\,(\ldots f\,(f\,(x_0, u_0), u_1) \ldots, u_{n-1})$$

be the nth iteration of the recursion function f. In this notation, when the system starts at an initial condition x_0 and is driven by an input list $u_0, \ldots, u_{n-1}$, the state it reaches at the nth step is given by

$$x_n = f^{\,n}(x_0, u_0, \ldots, u_{n-1})$$

Then, in crude terms, the realization $x_{k+1} = f\,(x_k, u_k)$ is reachable if the function $f^{\,n}$ is an open function. This definition reduces to the standard notion of reachability in the linear case.

Using this notion of reachability, we develop in section 4 a general computational framework for the calculation of permissible disturbance amplitudes and appropriate state feedback controllers for nonlinear systems. The resulting computational framework relies on the solution of a set of algebraic inequalities, derived directly from the given recursion function f of the system Σ.

The discussion presented here builds on results presented in Hammer (1989b, 1991). Alternative studies on the global control of nonlinear systems are given in Hammer (1984a,b, 1985, 1989a, 1994), Desoer and Kabuli (1988), Verma (1988), Sontag (1989), Chen and de Figueiredo (1990), Paice and Moore (1990), Verma and Hunt (1993), Paice and van der Schaft (1994), Baramov and Kimura (1995) and the references cited in these and other papers.

2. Preliminaries

The present section introduces the basic notation and set-up. Let R denote the set of all real numbers, and let R^m be the set of all m-dimensional real vectors, where m is a positive integer. Denote by $S(R^m)$ the set of all sequences $u_0, u_1, u_2, \dots$ of real vectors $u_i \in R^m$, $i = 0, 1, 2, \dots$ It is convenient to use the letter u to denote the sequence $u_0, u_1, u_2, \dots$ Then, u_i is the ith element of the sequence u.

Our discussion relates to systems Σ that are given in terms of a state representation of the form (1). Here, $u = u_0, u_1, u_2, \dots \in S(R^m)$ is the input sequence of Σ, and $x = x_0, x_1, x_2, \dots \in S(R^n)$ is the sequence of states through which the system Σ passes. The initial condition of the system is then x_0. The recursion function $f: R^n \times R^m \to R^n$ of Σ is a continuous function. In the present paper we restrict our attention to time-invariant systems, namely, to recursion functions f that do not depend directly on the step counter k. However, many aspects of our discussion can be directly generalized to time-varying recursion functions. It will be convenient to denote by $\Sigma(x_0)$ the response of the system Σ from the initial condition x_0.

The main topic of our discussion relates to the effects of disturbances on nonlinear control systems. In order to describe the magnitude of disturbances or of their effects, we shall use the standard l^∞-norm, which is defined as follows. Given a vector $v \in R^m$ with the components $(v^1, \dots, v^m)$, denote by

$$|v| := \max_{i=1,2,\dots,m} |v^i|$$

the maximal absolute value of a component. For a sequence $u = (u_0, u_1, \dots) \in S(R^m)$, set

$$|u| := \sup_{i \geq 0} |u_i|$$

so that $|u|$ is the standard l^∞-norm of the sequence u.

For a real number $\theta > 0$, we denote by $[-\theta, \theta]^m$ the set of all vectors $v \in R^m$ satisfying $|v| \leq \theta$, and by $S(\theta^m)$ the set of all vector sequences $u \in S(R^m)$ satisfying $|u| \leq \theta$.

According to figure 1 and (2), there are four disturbance signals that affect our configuration, namely, the signals υ_1, υ_2, υ_3, and υ_4. Of these, two disturbance signals (υ_2 and υ_3) affect the state value, and two (υ_1 and υ_4) affect the input value. It will be convenient to combine υ_1 and υ_4 into one total disturbance that acts on the system input, by defining

$$\upsilon_u := \upsilon_1 + \upsilon_4$$

We then impose an amplitude bound on the total input disturbance by requiring $|\upsilon_u| \leq \delta$.

The disturbance signals υ_2 and υ_3 of (2) affect the state value of the system Σ. As it turns out, the disturbance υ_2 representing the modelling errors, has a double effect; intuitively spreaking, this occurs since υ_2 affects both the value of the recursion (2) and the value of the state feedback function σ. In this sense, the total disturbance amplitude relating to the state x is in fact equivalent to $2|\upsilon_2| + |\upsilon_3|$. Allowing each of these two disturbance sources the same amplitude, and requiring the combined effect not to exceed δ, we restrict $|\upsilon_2| \leq \delta/3$ and $|\upsilon_3| \leq \delta/3$, so that $2|\upsilon_2| + |\upsilon_3| \leq \delta$. We can then summarize the combined disturbance restrictions in the form

$$\begin{aligned} |\upsilon_1 + \upsilon_4| &\leq \delta \\ |\upsilon_2 + \upsilon_3| &\leq 2\delta/3 \end{aligned} \tag{5}$$

Our objective is to find the largest value δ_m of the real number $\delta > 0$ that is compatible with the design specifications. The largest uniform bound on the amplitudes of the individual disturbances is then $d = \delta_m/3$ (although for the input disturbances v_1 and v_4, amplitudes up to $\delta_m/2$ are allowed). For the sake of simplifying the terminology, the term 'disturbance amplitude' will refer in the sequel to the number δ, rather than to the individual amplitudes of the disturbances v_1, v_2, v_3, or v_4.

The term 'nominal' is used below to refer to the response of the configuration (figure 1) when all disturbances are set to zero, and the external input sequence v is the zero sequence. We denote by $\Sigma_\sigma(x_0)$ the nominal response of the closed loop system when started at the initial condition $x_0 \in R^n$. When disturbances v_1, v_2, v_3 or v_4 are present, it is sometimes convenient to denote the response of the closed loop system by $\Sigma_\sigma(x_0) * (v_1, v_2, v_3, v_4)$; the initial condition here is still x_0, and the external input sequence is still $v = 0$. Of course, the exact value of the output sequence y in this case depends on the particular values of the disturbance sequences v_1, v_2, v_3, and v_4.

We conclude this section with a further note on notation. Throughout our discussion we shall need to study subsets of the cross product space $R^n \times R^m$. As usual, given a function $f: R^n \times R^m \to R^n$ and a subset $S \subset R^n \times R^m$, we shall denote by $f[S]$ the image of S through the function f. Thus, $f[S]$ is the set of all values $f(x, u)$ for pairs $(x, u) \in S$. Given two subsets $X \subset R^n$ and $U \subset R^m$, it will be convenient to denote by (X, U) the cross product set $X \times U$, so that

$$(X, U) = \{(x, u): x \in X \quad \text{and} \quad u \in U\}$$

Then, $f(X, U)$ is simply the subset of R^n consisting of all values $f(x, u)$ where $x \in X$ and $u \in U$.

3. State feedback

We turn now to a more detailed examination of the control configuration (figure 1). Here, Σ is the system that needs to be controlled. Its nominal model is given by (1), while its model with the disturbances active is represented by (2). The feedback is created by the feedback function $\sigma: R^n \to R^m$: $s_k = \sigma(y_k)$, $k = 0, 1, 2, \ldots$ With no disturbances present, we have $y_k = x_k$, $k = 0, 1, 2, \ldots$, and the input sequence u of Σ satisfies $u_k = v_k + s_k$, so that $x_{k+1} = f(x_k, v_k + s_k)$. Thus, without disturbances, the closed loop system Σ_σ is described by the recursion

$$x_{k+1} = f(x_k, (\sigma(x_k) + v_k)), \quad k = 0, 1, 2, \ldots$$

With all disturbances active, the recursive representation of the closed loop system Σ_σ takes the form

$$x_{k+1} = f(x_k, (\sigma(x_k + v_{3k}) + v_k + v_{1k} + v_{4k})) + v_{2k}, \quad k = 0, 1, 2, \ldots \tag{6}$$

Recall that our objective is to find a feedback function σ that drives the system Σ so that the output of the closed loop system Σ_σ does not deviate by more than $\Delta > 0$ from the zero output sequence. This has to be achieved over a range of initial conditions and disturbance signals, while the nominal external reference sequence v is set to zero. In specific terms, the problem can be stated as follows.

Design problem: *Given a pair of real numbers ρ, $\Delta > 0$, find the largest real number $\delta > 0$ (if one exists) for which there is a feedback function σ: $R^n \to R^m$ satisfying*

$$|y_{k+1}| = |f(x_k, (\sigma(x_k + v_{3k}) + v_{1k} + v_{4k})) + v_{2k} + v_{3k+1}| \leq \Delta, \quad k = 0, 1, 2, \ldots \tag{7}$$

for all disturbance signals satisfying $|v_1| \le \delta/2$, $|v_4| \le \delta/2$, $|v_2| \le \delta/3$, *and* $|v_3| \le \delta/3$, *and for all initial conditions* x_0 *with* $|x_0| \le \rho$. *For this value of* δ, *find an appropriate state feedback function* σ.

The number ρ represents here a permissible magnitude range for the initial condition x_0. To comply with the requirement $|y_0| \le \Delta$, while permitting a total discrepancy of magnitude δ between the initial condition x_0 and the initial output value y_0, we restrict $\rho \le \Delta - \delta$.

The main results of the present paper are the derivation of necessary and sufficient conditions for the existence of appropriate state feedback functions σ; the development of computational techniques for the calculation of appropriate state feedback functions σ; and the characterization of the maximal permissible disturbance amplitude bound δ. We turn now to an examination of some concepts that underlie our discussion.

3.1. *Static state feedback and eigensets*

In this subsystem we review and refine the notion of an eigenset (introduced in Hammer (1989b)), and we point out the relation between eigensets and the control problem at hand. We start with some notation. Given a real number $\delta > 0$ and a vector $x \in R^q$, denote by $B(\delta, x)$ the ball of radius δ in R^q that is centred at the point x, namely,

$$B(\delta, x) := \{z \in R^q \colon |z - x| \le \delta\}$$

Note that since we are using the l^∞-norm, a ball is in fact a rectangular cube. The term 'the open ball $B(\delta, x)$' refers to the set $B(\delta, x) := \{z \in R^q \colon |z - x| < \delta\}$.

Next, for a subset $S \subset R^q$, denote by $N_\delta(S)$ the 'δ-neighbourhood' of S in R^q, i.e. the set

$$N_\delta(S) := \bigcup_{x \in S} B(\delta, x)$$

Of course, when the set S consists of a single point $x \in R^q$, we have $N_\delta(x) = B(\delta, x)$.

The radius $|S|$ of a subset $S \subset R^q$ is the radius of the smallest ball around the origin that contains S, and is given by

$$|S| := \sup_{x \in S} |x|$$

Given a pair of subsets $S_x \subset R^n$ and $S_u \subset R^m$, and a function $f \colon R^n \times R^m \to R^n$, we denote by $f[S_x, S_u]$ the image of the cross product set $S_x \times S_u$ through f, namely,

$$f[S_x, S_u] = \{f(x, u) \colon x \in S_x, u \in S_u\}$$

Note that $f[S_x, S_u]$ is a subset of R^n.

Finally, denote by $\Pi_x : R^n \times R^m \to R^n : (x, u) \mapsto x$ the standard projection onto the first n coordinates.

We can now define the basic concept on which our discussion is based. This concept is a slight variant of the concept of an eigenset introduced in Hammer (1989b).

Definition 1: *Let* $f \colon R^n \times R^m \to R^n$ *be a function, and let* $\delta, \Delta > 0$ *be a pair of real numbers, where* $\delta \le \Delta$. *A non-empty subset* $S \subset R^n \times R^m$ *is a* (δ, Δ)-eigenset *of the function* f *if it satisfies the following conditions:*

(i) $|\Pi_x S| \leq \Delta - \delta$
(ii) $f[N_\delta(S)] \subset \Pi_x S$

The number Δ is called the bound *of the eigenset S, whereas δ is called the* contraction value.

Property (ii) of Definition 1 means that S is a conditional invariant subset of the function f. Furthermore, deviations of magnitude not exceeding δ in the state x, as well as in the input value u, do not destroy this conditional invariance property. The notation of an eigenset is a refinement of the classical concepts of invariant subset and conditional invariant subset, which have played important roles in the evolution of nonlinear as well as linear system theory (e.g. Lasalle and Lefschetz 1961, Lefschetz 1965, Wonham 1974). Before discussing the calculation of (δ, Δ)-eigensets, we discuss their significance to the control problem at hand.

Let $S \subset R^n \times R^m$ be a (δ, Δ)-eigenset of the recursion function $f: R^n \times R^m \to R^n$. With each state $x \in N_\delta(\Pi_x S)$, we associate a set of input values $U(x, S) \subset R^m$ for which the next state of the system is in $\Pi_x S$ as follows.

(i) For a state $x \in \Pi_x S$, the set $U(x, S)$ consists of all vectors $u \in R^m$ for which $(x, u) \in S$.
(ii) For a state $x \notin \Pi_x S$, we distinguish between the following two cases.
  (a) When $x \in N_{2\delta/3}(\Pi_x S)$, then $U(x, S)$ consists of all vectors $u \in R^m$ for which $(y, u) \in S$ for some $y \in R^n$ satisfying $|y - x| \leq 2\delta/3$.
  (b) When $x \notin N_{2\delta/3}(\Pi_x S)$, then $U(x, S)$ consists of all vectors $u \in R^m$ for which $(y, u) \in S$ for some vector $y \in R^n$ satisfying $|y - x| \leq \delta$.

The set $U(x, S)$ is critical to our discussion, and we list now a few of its technical features.

Lemma 1: *The set $U(x, S)$ has the following properties.*

(i) $U(x, S) \neq \emptyset$ for all $x \in N_\delta(\Pi_x S)$.
(ii) $(x, N_\delta[U(x, S)]) \subset N_\delta(S)$ for every vector $x \in N_\delta(\Pi_x S)$.
(iii) $(z, N_\delta[U(x, S)]) \subset N_\delta(S)$ for all vectors $x, z \in N_{2\delta/3}(\Pi_x S)$ satisfying $|z - x| \leq \delta/3$.

Proof: Part (i) follows directly from the construction of the set $U(x, S)$, combined with the fact that $S \neq \emptyset$ according to Definition 1.

Next, regarding (ii), consider a point $x \in N_\delta(\Pi_x S)$, and let $u \in U(x, S)$ be an input value. Then, by construction of $U(x, S)$, there is a vector $y \in N_\delta(x)$ such that $(y, u) \in S$. But then clearly $N_\delta(y, u) \subset N_\delta(S)$; since $N_\delta(y, u) = (N_\delta(y), N_\delta(u))$ and $x \in N_\delta(y)$, it follows that $(x, N_\delta(u)) \subset N_\delta(S)$. The latter holds for all $u \in U(x, S)$, and therefore $(x, N_\delta[U(x, S)]) \subset N_\delta(S)$, as required.

Finally, in order to prove (iii), consider a point $x \in N_{2\delta/3}(\Pi_x S)$, and let $u \in U(x, S)$ be an input value. Then, by construction of $U(x, S)$, there is a vector $y \in N_{2\delta/3}(x)$ such that $(y, u) \in S$. As in the previous paragraph, this implies that $(N_\delta(y), N_\delta(u)) \subset N_\delta(S)$. Now, $|z - y| = |(z - x) - (y - x)| \leq |z - x| + |y - x| \leq \delta/3 + 2\delta/3$, where we have used the Lemma assumption $|z - x| \leq \delta/3$ combined with the fact that $y \in N_{2\delta/3}(x)$. Thus, $|z - y| \leq \delta$, so that $z \in N_\delta(y)$, and we obtain that $(z, N_\delta(u)) \subset N_\delta(S)$. Since this holds for all $u \in U(x, S)$, we obtain $(z, N_\delta[U(x, S)]) \subset N_\delta(S)$, and our proof is complete. $\square$

Consider a system Σ: $S(R^m) \to S(R^n)$ having the recursive representation $x_{k+1} = f(x_k, u_k)$. Assume the recursion function f has a (δ, Δ)-eigenset S, and that the system is started from an initial condition $x_0 \in \Pi_x S$. We can construct an input sequence $u = (u_0, u_1, \ldots) \in S(R^m)$ for Σ that drives Σ so that all the states along the resulting path in state space belong to $\Pi_x S$. Indeed, using the fact that $x_0 \in \Pi_x S$, we construct the input sequence u recursively as follows: whenever $x_i \in \Pi_x S$, take an input value

$$u_i \in U(x_i, S) \tag{8}$$

This yields $x_{i+1} = f(x_i, u_i) \in \Pi_x S$. Since $x_0 \in S$, this implies $x_i \in \Pi_x S$ for all integers $i \geq 0$, and the required input sequence u is obtained. Note that since $|\Pi_x S| \leq \Delta - \delta$, the sequence x of states also satisfies $|x + v| \leq \Delta$ for any disturbance v of amplitude not exceeding δ.

The input sequence u constructed according to (8) has a property that is critical to our discussion: at each step $i \geq 0$, the input value u_i is assigned based on the value x_i of the state at that step. In other words, the input value is assigned through a state feedback mechanism.

The assignment (8) yields a general methodology for the design of static state feedback controllers for nonlinear control systems, as pointed out in Hammer (1989b). Our present situation, however, is somewhat simpler than that of Hammer (1989b), since presently we do not require the feedback function σ to be a continuous function. When the continuity requirement on the state feedback function σ is released, the main result of Hammer (1989b) simplifies into the following form.

Theorem 1: *Let Σ be a system with the recursion function $f: R^n \times R^m \to R^n$, and assume f has a (δ, Δ)-eigenset S for some real numbers δ, $\Delta > 0$. Define a state feedback function σ: $R^n \to R^m$ as follows.*

(i) *For a state $x \in N_\delta(\Pi_x S)$, set $\sigma(x) := u$, where u is any element of the set $U(x, S)$.*

(ii) *For all other states $x \in R^n$, set $\sigma(x) := 0$.*

Then, for any initial condition $x_0 \in \Pi_x S$ and for any disturbances satisfying $|v_1| \leq \delta/2$, $|v_4| \leq \delta/2$, $|v_2| \leq \delta/3$, and $|v_3| \leq \delta/3$, the closed loop system satisfies $|\Sigma_\sigma(x_0)^*(v_1, v_2, v_3, v_4)| \leq \Delta$.

Proof: Let S be a (δ, Δ)-eigenset of the recursion function f of Σ. Then, by definition, $f[N_\delta(S)] \subset \Pi_x S$ and $|\Pi_s S| \leq \Delta - \delta$.

Consider now a state $\xi \in \Pi_x S$, and, referring to (6) and (7), consider the point $(\xi + v_0, (\sigma(\xi + v_0 + v_{3,0}) + v_{1,0} + v_{4,0}))$, where an extra disturbance v_0 has been added to the state ξ, and where the amplitudes satisfy $|v_0| \leq \delta/3$, $|v_{3,0}| \leq \delta/3$ and $|v_{1,0} + v_{4,0}| \leq \delta$. Setting $z := \xi + v_0$ and $x := \xi + v_0 + v_{3,0}$, we obtain $|z - \xi| = |v_0| \leq \delta/3$, $|x - \xi| \leq |v_0 + v_{3,0}| \leq 2\delta/3$, and $|z - x| = |v_{3,0}| \leq \delta/3$. The first two inequalities imply that x, $z \in N_{2\delta/3}(\Pi_x S)$, and whence, invoking Lemma 1 (iii), it follows that $(z, N_\delta(U(x, S))) \subset N_\delta(S)$. But then, the definition of σ combined with the fact that $|v_{1,0} + v_{4,0}| \leq \delta$ directly yields that $(\xi + v_0, (\sigma(\xi + v_0 + v_{3,0}) + v_{1,0} + v_{4,0})) \in N_\delta(S)$. Since S is a (δ, Δ)-eigenset of f, we obtain that

$$f(\xi + v_0, (\sigma(\xi + v_0 + v_{3,0}) + v_{1,0} + v_{4,0})) \in \Pi_x S \tag{9}$$

Now, set $v_0 = 0$, $\xi := x_0$ and $\xi_1 := f(x_0, (\sigma(x_0 + v_{3,0}) + v_{1,0} + v_{4,0}))$. Then, since $x_0 \in \Pi_x S$ by the assumption of Theorem 1, (9) yields $\xi_1 \in \Pi_x S$. Since the first output value y_1 satisfies

$$y_1 = \xi_1 + v_{2,0} + v_{3,0}$$

we have

$$|y_1| = |\xi_1 + v_{2,0} + v_{3,0}| \leq |\Pi_x S| + |v_{2,0} + v_{3,0}| \leq \Delta - \delta + 2\delta/3 < \Delta$$

as required.

In preparation for an induction, assume the vector $\xi_i \in \Pi_x S$ has been obtained for some integer $i \geq 0$, and set

$$\xi_{i+1} := f(\xi_i + v_{2,i-1}, (\sigma(\xi_i + v_{2,i-1} + v_{3,i}) + v_{1,i} + v_{4,i}))$$

It then follows from (9) that $\xi_{i+1} \in \Pi_x S$ for all disturbances satisfying $|v_1| \leq \delta/2$, $|v_4| \leq \delta/2$, $|v_2| \leq \delta/3$, and $|v_3| \leq \delta/3$, where $v_{2,-1} := 0$. In particular, this implies that $|\xi_{i+1}| \leq \Delta - \delta$. The output value of the system then satisfies

$$y_{i+1} = \xi_{i+1} + v_{2,i} + v_{3,i+1}$$

so that

$$|y_{i+1}| = |\xi_{i+1} + v_{2,i} + v_{3,i+1}| \leq |\xi_{i+1}| + |v_{2,i} + v_{3,i+1}| \leq \Delta - \delta + 2\delta/3 < \Delta$$

This completes our proof. □

Thus we see that a (δ, Δ)-eigenset of the recursion function f gives rise to state feedback functions σ that satisfy our design requirements. Note that for this to be valid, the system Σ must start from an initial condition $x_0 \in \Pi_x S$. It is usually desirable in applications to have a range of permissible initial conditions, namely, that the set of permissible initial conditions contain a ball $B(\rho, 0)$ of some radius $\rho > 0$. This requirement has been incorporated into the statement of Design problem (7). From our present discussion, it leads to the condition $B(\rho, 0) \subset \Pi_x S$. We continue now with our investigation of general properties of eigensets.

A slight reflection shows that the union of (δ, Δ)-eigensets of the function f is again a (δ, Δ)-eigenset of the same function f. This implies that there is a maximal (δ, Δ)-eigenset $S(\delta, \Delta) \subset R^n \times R^m$ of the function f, given by the union

$$S(\delta, \Delta) := \{\cup S\colon S \text{ is a } (\delta, \Delta)\text{-eigenset of } f\} \tag{10}$$

Recall that by definition, a (δ, Δ)-eigenset is not empty, so that there are recursion functions f that do not possess (δ, Δ)-eigensets. We discuss later general conditions on the function f that guarantee the existence of (δ, Δ)-eigensets (see in particular Theorem 4 below).

We mention next a property of the maximal (δ, Δ)-eigenset $S(\delta, \Delta)$ that helps enlighten its intuitive meaning. Consider a system Σ having the recursion function f. We have seen in Theorem 1 that the closed loop system Σ_σ satisfies the design requirements (7) when it is started from an initial condition x_0 that belongs to the projection $\Pi_x S(\delta, \Delta)$ onto the state space. In fact, as indicated by the next statement, more is true: the set $\Pi_x S(\delta, \Delta)$ consists of all initial conditions from which the system can tolerate disturbances of amplitude δ on its state and input variables, without exceeding the bound Δ with its state amplitudes.

Proposition 1: *Let Σ be a system with the recursion function f, and assume that f has a maximal (δ, Δ)-eigenset $S(\delta, \Delta)$. Then, $\Pi_x S(\delta, \Delta)$ consists of exactly all initial conditions $x_0 \in R^n$ for which the following holds: there is an input sequence $u \in S(R^m)$ for which the sequence $x \in S(R^n)$ given by the recursion*

$$x_{k+1} = f(x_k + v_k, u_k + \omega_k), \; k = 0, 1, 2, \ldots$$

satisfies $|x| \le \Delta - \delta$ for all pairs of sequences $v \in S(\delta^n)$ and $\omega \in S(\delta^m)$.

Proof: Consider first an initial condition $x_0 \in \Pi_x S(\delta, \Delta)$. In preparation for induction, let $k \ge 0$ be an integer for which $x_k \in \Pi_x S(\delta, \Delta)$. Then $|x_k| \le \Delta - \delta$, and there is a vector $u_k \in R^m$ such that $(x_k, u_k) \in S(\delta, \Delta)$. Further, since $v \in S(\delta^n)$ and $\omega \in S(\delta^m)$, we have that $(x_k + v_k, \; u_k + \omega_k) \in N_\delta(S(\delta, \Delta))$, and it follows that $x_{k+1} = f(x_k + v_k, u_k + \omega_k) \in f[N_\delta(S(\delta, \Delta))] \subset S(\delta, \Delta)$, so that $|x_{k+1}| \le \Delta - \delta$. Thus, by induction, for every initial condition $x_0 \in \Pi_x S(\delta, \Delta)$ there is an input sequence $u \in S(R^m)$ for which the state sequence x of Proposition 1 satisfies $|x| \le \Delta - \delta$.

Conversely, let $x_0 \in R^n$ be an initial condition, and assume there is an input sequence $u \in R^m$ for which the sequence $x(v, \omega)$ obtained by the recursion

$$x(v, \omega)_{k+1} = f(x(v, \omega)_k + v_k, u_k + \omega_k) \tag{11}$$

satisfies $|x(v, \omega)| \le \Delta - \delta$ for all pairs of sequences $v \in S(\delta^n)$ and $\omega \in S(\delta^m)$. Define the set

$$S := \bigcup_{k \ge 0, \, v \in S(\delta^n), \, \omega \in S(\delta^m)} (x(v, \omega)_k, u_k)$$

which is a subset of $R^n \times R^m$. It follows then directly from (3.1.8) that $f[N_\delta(S)] \subset \Pi_x S$ and $|\Pi_x S| \le \Delta - \delta$. Consequently, S is a (δ, Δ)-eigenset of f, and, in view of (10), we have $S \subset S(\delta, \Delta)$. Thus, $x_0 \in \Pi_x S(\delta, \Delta)$, and our proof concludes. □

3.2. *A characterization of eigensets*

Having discussed the control theoretic significance of (δ, Δ)-eigensets, we turn now to the derivation of a direct characterization of eigensets. We show that, in principle, the maximal (δ, Δ)-eigenset $S(\delta, \Delta)$ of the function $f: R^n \times R^m \to R^n$ is determined by the solution of a system of algebraic inequalities. In general, however, this system of inequalities is infinite. In the second half of this subsection and in section 4 we discuss techniques that yield (δ, Δ)-eigensets of the function f through the solution of a finite set of algebraic inequalities. These techniques, in combination with the construction of feedback functions described in Theorem 1, provide computational means for the derivation of state feedback functions that solve the design problem (7).

From this point on, it will be convenient to adopt the realistic assumption that the system Σ being controlled accepts only input values of amplitude not exceeding a specified bound $\mu > 0$. The value of the bound μ is usually determined by physical characteristics of the components of the system Σ. Then, only input values $u \in [-\mu, \mu]^m$ are permitted, and the recursion function f of Σ has the domain $R^n \times [-\mu, \mu]^m$. Let $S(\delta, \Delta, \mu)$ be the maximal (δ, Δ)-invariant subset of f, with input values bounded by μ.

For a function $f: R^n \times [-\mu, \mu]^m \to R^n$, let f^* denote the inverse set function, which maps subsets of R^n into subsets of $R^n \times [-\mu, \mu]^m$. Explicitly, for a subset $X \subset R^n$, the subset $f^*[X]$ consists of all pairs $(x, u) \in R^n \times [-\mu, \mu]^m$ satisfying $f(x, u) \in X$.

Now, let Σ be a system with the recursive representation $x_{k+1} = f(x_k, u_k)$, where $f: R^n \times [-\mu, \mu]^m \to R^n$ is a continuous function. Given a pair of real numbers Δ, $\delta > 0$, where $\delta \leq \Delta$, let $P(\delta, \Delta) \subset R^n \times [-\mu, \mu]^m$ be the set of all pairs (x, u), where $x \in R^n$ and $u \in [-\mu, \mu]^m$, for which the following hold:

$$\begin{aligned} &|x| \leq \Delta - \delta \\ &|f(y, v)| \leq \Delta - \delta \text{ for all } y, v \text{ satisfying } |y - x| \leq \delta, |v - u| \leq \delta \end{aligned} \tag{12}$$

Note that (12) is a set of algebraic inequalities based on the given recursion function f of Σ, and $P(\delta, \Delta)$ can be found by direct calculation. Note also that every point $(x, u) \in P(\delta, \Delta)$ must satisfy $|x| \leq \Delta - \delta$ and $|u| \leq \mu$, so that $P(\delta, \Delta)$ is a bounded set. The continuity of the function f implies that $P(\delta, \Delta)$ is a closed subset. Furthermore, since $P(\delta, \Delta)$ consists of all points $|x| \leq \Delta - \delta$ for which there is a $u \in [-\mu, \mu]^m$ such that $|f[N_\delta(x), N_\delta(u)]| \leq \Delta - \delta$, it follows directly that the maximal (Δ, δ)-eigenset $S(\delta, \Delta, \mu)$ of f is a subset of $P(\delta, \Delta)$.

Next, define a sequence $P_i(\delta, \Delta)$, $i = 0, 1, 2, \ldots$, of subsets of $R^n \times [-\mu, \mu]^m$ based on P, as follows.

$$\begin{aligned} P_0(\delta, \Delta) &= P(\delta, \Delta) \\ \Theta_{i+1} &= P_i(\delta, \Delta) \cap f^*[\Pi_x P_i(\delta, \Delta)] \\ P_{i+1}(\delta, \Delta) &= \{(x, u) \in \Theta_{i+1}: f[N_\delta(x, u)] \subset \Pi_x \Theta_{i+1}\} \end{aligned} \tag{13}$$

$i = 0, 1, 2, \ldots$ In particular, it is a direct consequence of (13) that the sets $\{P_i(\delta, \Delta)\}$ satisfy

$$f[N_\delta(P_{i+1}(\delta, \Delta))] \subset \Pi_x P_i(\delta, \Delta) \tag{14}$$

Furthermore, (13) implies directly that $\{P_i(\delta, \Delta)\}$ is a monotonic non-increasing sequence of sets, i.e. that $P_o(\delta, \Delta) \supset P_1(\delta, \Delta) \supset P_2(\delta, \Delta) \supset \ldots$ An inspection of (13) also shows that if there is an integer $k \geq 0$ for which $P_{k+1}(\delta, \Delta) = P_k(\delta, \Delta)$, then $P_i(\delta, \Delta) = P_k(\delta, \Delta)$ for all integers $i \geq k$. Also, combining our earlier observation that $P(\delta, \Delta)$ is a bounded and closed subset with the fact that f is a continuous function, it follows from (13) that all subsets $P_i(\delta, \Delta)$ are bounded and closed (i.e. compact) subsets.

Finally, define the intersection set

$$P_\infty(\delta, \Delta) := \bigcap_{i \geq 0} P_i(\delta, \Delta)$$

Then, the following is true.

Theorem 2: *Let δ, $\Delta > 0$ be two real numbers, where $\delta \leq \Delta$, and let $f: R^n \times [-\mu, \mu]^m \to R^n$ be a continuous function. Then, the maximal (δ, Δ)-eigenset of f is given by $S(\delta, \Delta, \mu) = P_\infty(\delta, \Delta)$, and f has no (δ, Δ)-eigensets when $P_\infty(\delta, \Delta) = \varnothing$.*

Proof: We first show that $S(\delta, \Delta, \mu) \subset P_\infty(\delta, \Delta)$. To this end, it has been indicated earlier that $S(\delta, \Delta, \mu) \subset P_0(\delta, \Delta)(= P(\delta, \Delta))$. Also, since $S(\delta, \Delta, \mu)$ is a (δ, Δ)-eigenset of f, we have

$$f[N_\delta(S(\delta, \Delta, \mu))] \subset \Pi_x S(\delta, \Delta, \mu) \tag{15}$$

which implies that

$$S(\delta, \Delta, \mu) \subset f^*[\Pi_x S(\delta, \Delta, \mu)] \tag{16}$$

Now, if $S(\delta, \Delta, \mu) \subset P_i(\delta, \Delta)$ for some integer $i \geq 0$, it follows from (15), (16), and (13) that $S(\delta, \Delta, \mu) \subset P_{i+1}(\delta, \Delta)$. Since $S(\delta, \Delta, \mu) \subset P_0(\delta, \Delta)$, this implies that $S(\delta, \Delta, \mu) \subset P_i(\delta, \Delta)$ for all integers $i \geq 0$, so that $S(\delta, \Delta, \mu) \subset P_\infty(\delta, \Delta)$.

Next, we show that $P_\infty(\delta, \Delta) \subset S(\delta, \Delta, \mu)$. Note that by (13), we have $N_\delta(P_{i+1}(\delta, \Delta)) \subset P_i(\delta, \Delta) \cap f^*[\Pi_x P_i(\delta, \Delta)]$. Then, starting with the definition of $P_\infty(\delta, \Delta)$,

$$\begin{aligned} f[N_\delta(P_\infty(\delta, \Delta))] &= f\left[N_\delta\left(\bigcap_{i\geq 0} P_{i+1}(\delta, \Delta)\right)\right] \\ &\subset f\left[\bigcap_{i\geq 0} N_\delta(P_{i+1}(\delta, \Delta))\right] \\ &\subset \bigcap_{i\geq 0} f[N_\delta(P_{i+1}(\delta, \Delta))] \\ &\subset \bigcap_{i\geq 0} \Pi_x P_i(\delta, \Delta) \\ &= \Pi_x\left[\bigcap_{i\geq 0} P_i(\delta, \Delta)\right] \end{aligned}$$

where the step before last is a consequence of (14), and the last equality is a consequence of the fact that $\{P_i(\delta, \Delta)\}$ is a monotonic decreasing sequence of compact sets. But the last term is simply $\Pi_x P_\infty(\delta, \Delta)$, so we obtain

$$f[N_\delta(P_\infty(\delta, \Delta))] \subset \Pi_x P_\infty(\delta, \Delta)$$

Finally, since $P_\infty(\delta, \Delta) \subset P(\delta, \Delta)$ and $P(\delta, \Delta)$ is given by (12), it follows that $P_\infty(\delta, \Delta)$ is a (δ, Δ)-eigenset of the recursion function f. In view of (10), this implies that $P_\infty(\delta, \Delta) \subset S(\delta, \Delta, \mu)$. Thus, combining with the first part of the proof, we have $P_\infty(\delta, \Delta) = S(\delta, \Delta, \mu)$, as required. □

The characterization of the maximal (δ, Δ)-eigenset $S(\delta, \Delta, \mu)$ given by Theorem 2 involves, in general, an infinite computational process. Still, one can derive from this characterization a finite process that yields an eigenset of f that is an 'approximation' of $S(\delta, \Delta, \mu)$, as follows.

Corollary 1: Let $f: R^n \times [-\mu, \mu]^m \to R^n$ *be a continuous function, and let* δ, Δ, $\varepsilon > 0$ *be real numbers, where* $\delta + \varepsilon < \Delta$. *Assume* f *has a* $(\delta + \varepsilon, \Delta)$*-eigenset. Then, there is an integer* $n(\varepsilon) \geq 0$ *such that* $P_{n(\varepsilon)}(\delta + \varepsilon, \Delta)$ *is a* (δ, Δ)*-eigenset of* f.

Proof: Recall that the sequence $\{P_i(\delta + \varepsilon, \Delta)\}$ is a monotone non-increasing sequence of compact subsets of $R^n \times [-\mu, \mu]^m$. In view of Theorem 2 and the fact that f has a $(\delta + \varepsilon, \Delta)$-eigenset, it follows that $P_i(\delta + \varepsilon, \Delta) \neq \varnothing$ for all $i \geq 0$.

Let $S_1 \subset S_2 \subset R^q$ be two compact subsets. For each point $x \in S_2$, let $d(x)$ be the distance to the closest point in S_1, i.e.

$$d(x) := \inf\{|x - y| : y \in S_1\}$$

Then, define the *discrepancy* $d(S_1, S_2)$ as the largest distance between a point in S_2 and its closest neighbour in S_1, i.e.

$$d(S_1, S_2) := \sup\{d(x) \colon x \in S_2\}$$

Now, for each integer $i \geq 0$, let

$$d_i := d(P_\infty(\delta + \varepsilon, \varDelta),\ P_i(\delta + \varepsilon, \varDelta))$$

Since $\{P_i(\delta + \varepsilon, \varDelta)\}$ is a monotone non-increasing sequence of compact sets having the limit $P_\infty(\delta + \varepsilon, \varDelta)$, it follows that the sequence $\{d_i\}$ of real non-negative numbers must also be monotone non-increasing. Consequently, it has a limit, say

$$d := \lim_{i \to \infty} d_i$$

Since $P_\infty(\delta + \varepsilon, \varDelta) = \bigcap_{i \geq 0} P_i(\delta + \varepsilon, \varDelta)$ is the intersection of the monotone non-increasing sequence of compact sets, it follows that $d = 0$. There is then an integer $n(\varepsilon) \geq 0$ such that $d_j < \varepsilon$ for all $j \geq n(\varepsilon)$, which implies

$$N_\delta(P_{n(\varepsilon)}(\delta + \varepsilon, \varDelta)) \subset N_{\delta+\varepsilon}(P_\infty(\delta + \varepsilon, \varDelta))$$

Taking into account the fact that $f[N_{\delta+\varepsilon}(P_\infty(\delta + \varepsilon, \varDelta))] \subset \Pi_x P_\infty(\delta + \varepsilon, \varDelta)$ by Theorem 2, it follows that

$$f[N_\delta(P_{n(\varepsilon)}(\delta + \varepsilon, \varDelta))] \subset f[N_{\delta+\varepsilon}(P_\infty(\delta + \varepsilon, \varDelta))] \subset \Pi_x P_\infty(\delta + \varepsilon, \varDelta).$$

Since $P_\infty(\delta + \varepsilon, \varDelta) \subset P_{n(\varepsilon)}(\delta + \varepsilon, \varDelta)$, we obtain

$$f[N_\delta(P_{n(\varepsilon)}(\delta + \varepsilon, \varDelta))] \subset \Pi_x P_{n(\varepsilon)}(\delta + \varepsilon, \varDelta)$$

which, combined with (12), shows that $P_{n(\varepsilon)}(\delta + \varepsilon, \varDelta)$ is a $(\delta, \varDelta)$-eigenset of f. □

In intuitive terms, Corollary 1 shows that one can obtain a $(\delta, \varDelta)$-eigenset of the function f in a finite number of steps. This can be achieved by performing the first $n(\varepsilon)$ steps of the process of calculating the maximal eigenset of f with respect to the somewhat larger contraction radius $\delta + \varepsilon$. In practice, however, the value of $n(\varepsilon)$ may not be known in advance. In the next section we revisit the question of providing a finite technique for the calculation of $(\delta, \varDelta)$-eigensets. Using the notion of reachability, we develop there an explicit finite algorithm for the calculation of $(\delta, \varDelta)$-eigensets. In the meantime, we continue with our general discussion of eigensets.

Let Σ be a system represented by the recursion $x_{k+1} = f(x_k, u_k)$, where $f \colon R^n \times [-\mu, \mu]^m \to R^n$ is a continuous function, and assume that f has a $(\delta, \varDelta)$-eigenset for some real number $\delta > 0$. It follows then from Theorem 1 that one can construct a static state feedback controller σ so that the closed loop system Σ_σ tolerates disturbances of amplitude not exceeding δ. This fact can be used to characterize the maximal disturbance amplitude for which the design problem (7) can be solved, as follows. Let $S(\delta, \varDelta, \mu)$ be the maximal $(\delta, \varDelta)$-eigenset of f, set $S(\delta, \varDelta, \mu) := \varnothing$ when f has no $(\delta, \varDelta)$-eigensets, and assume there is a value $\delta > 0$ for which $S(\delta, \varDelta, \mu) \neq \varnothing$. Define the real number $\delta_{\mathrm{M}} > 0$ by the relation

$$\delta_{\mathrm{M}} := \sup\{\delta \colon S(\delta, \varDelta, \mu) \neq \varnothing\}$$

If there is no $\delta > 0$ for which $S(\delta, \varDelta) \neq \varnothing$, set $\delta_{\mathrm{M}} := 0$. We call δ_{M} the *eigenset contraction bound* of the function f, relative to $\varDelta$. In view of the fact that f is a continuous function, part (i) of the following statement is a consequence of Theorem 1; part (ii) follows from Proposition 1.

Corollary 2: *Let Σ: $S(\mu^m) \to S(R^n)$ be a system with the recursive representation $x_{k+1} = f(x_k, u_k)$, where $f: R^n \times [-\mu, \mu]^m \to R^n$ is a continuous function. Assume f has an eigenset contraction bound $\delta_M > 0$ relative to Δ. Then, the following are true.*

(i) *There is a feedback function σ: $R^n \to R^m$ that satisfies (7) with $\delta := \delta_M$.*

(ii) *If there is a real number $\delta > 0$ and an input sequence $u \in S(\mu^m)$ for which the state sequence $x \in S(R^n)$ given by the recursion $x_{k+1} = f(x_k + v_k, u_k + \omega_k)$, $k = 0, 1, 2, \ldots$, satisfies $|x + v| \leq \Delta$ for all disturbance sequences $v \in S(\delta^n)$ and $\omega \in S(\delta^m)$, then $\delta \leq \delta_M$.*

Corollary 2 shows that, in fact, δ_M characterizes the largest disturbance amplitude that our system can tolerate.

To summarize, the theory of eigensets allows us to develop a methodology for the design of nonlinear state feedback controllers, which is entirely based on characteristics of the known recursion function f of the system Σ being controlled. It is therefore important to develop effective techniques for the calculation of eigensets. In the next section we show that one can exploit the notion of reachability to obtain such a technique.

4. Reachability and eigensets

In the present section we show that the notion of reachability is instrumental for the calculation of eigensets in the general nonlinear case. This result sheds light on the connection between reachability and the problem of disturbance handling in control theory. First, of course, we have to clarify what is meant by the term 'reachability' in the nonlinear case, as numerous definitions have been used in the literature. The definition employed here is a direct adaptation of the linear notion of reachability.

4.1. *Local and global reachability*

Let Σ be a system described by the recursive representation

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, 2, \ldots \tag{17}$$

where $x_k \in R^n$ and $u_k \in R^m$, and consider the behaviour of Σ for the first i steps, where $i \geq 1$ is an integer. Assume the system is started from the initial condition $x_0 = x \in R^n$, and is driven by the input list $u_0, u_1, \ldots, u_{i-1}$. The states x_k, $k = 1, \ldots, i$, through which the system passes can be calculated recursively; we have $x_1 = f(x, u_0)$, $x_2 = f(x_1, u_1) = f(f(x, u_0), u_1), \ldots$, and, in general,

$$x_i = f(f \ldots f(f(x, u_0), u_1), \ldots, u_{i-1})$$

where the recursion function f is iterated i times. It is convenient to use the following shorthand notation for this iteration

$$f^i(x, u_0, \ldots, u_{i-1}) := f(f \ldots f(f(x, u_0), u_1), \ldots, u_{i-1})$$

so that $x_i = f^i(x, u_0, \ldots, u_{i-1})$.

In some instances we shall be interested in properties of the function f^i for a fixed value of the initial state x. In such cases, we shall consider the partial function $f^i(x,\cdot)$: $(R^m)^i \to R^n$: $(u_0,\ldots,u_{i-1}) \mapsto f^i(x,u_0,\ldots,u_{i-1})$.

We say that a state $x' \in R^n$ is *reachable* from the state $x \in R^n$ in i steps if there is an input list $u_0,\ldots,u_{i-1}$ for which $f^i(x,u_0,\ldots,u_{i-1}) = x'$. In other words, x' is reachable from x whenever x' is an element of the image of the partial function $f^i(x,\cdot)$. We denote this image by $\operatorname{Im} f^i(x,\cdot)$, i.e.

$$\operatorname{Im} f^i(x,\cdot) := \{f^i(x,u_0,\ldots,u_{i-1}): u_0,\ldots,u_{i-1} \in R^m\}$$

Recall that a function g: $R^q \to R^n$ is an *open function* if it maps every open subset of R^q onto an open subset of R^n; explicitly, for every open subset $S \subset R^q$, the image $g[S]$ is an open subset of R^n.

Definition 2: *The realization* (17) *is* everywhere locally reachable *if there is an integer $p \geq 1$ for which the function $f^p(x,\cdot)$ is an open function for all states $x \in R^n$.*

The following statement provides an example of a common class of recursion functions f that induce realizations that are everywhere locally reachable. For a proof, see e.g. the work of Buck (1978, Chapter 7, in particular the generalizations of Theorem 15 therein). Note that in this case, the integer p of Definition 2 is taken as $p = n$, where n is the dimension of the state vector x_k.

Proposition 2: *Assume that the recursion function f of* (17) *is continuously differentiable. For a point $(x,u_0,\ldots,u_{n-1}) \in R^n \times (R^m)^n$, define the $n \times (mn)$ matrix*

$$C(x,u_0,\ldots,u_{n-1}) := \frac{\partial f^n(x,u_0,\ldots,u_{n-1})}{\partial(u_0,\ldots,u_{n-1})} \tag{18}$$

If the matrix $C(x,u_0,\ldots,u_{n-1})$ is of full rank at all points $(x,u_0,\ldots,u_{n-1}) \in R^n \times (R^m)^n$, then the realization (17) *is everywhere locally reachable.*

In the particular case where f is a linear function of the form

$$f(x,u) = Au + Bu$$

where A and B are constant matrices, the matrix $C(x,u_0,\ldots,u_{n-1})$ of Proposition 2 is a constant matrix, equal to the controllability matrix of the realization. Thus, in the linear case, Proposition 2 reduces to the well known characterization of reachability.

Consider a realization (17) that is everywhere locally reachable. Then, by definition, there is an integer p for which the partial function $f^p(x,\cdot)$ is an open function for all $x \in R^n$. We denote by η the smallest possible value of the positive integer p, and we call η the *reachability integer* of the recursion function f.

Assume then that the realization (17) is everywhere locally reachable, with the reachability integer η. We say that this realization is *globally reachable* if every state $x' \in R^n$ is reachable from every state $x \in R^n$ in η steps.

Let Σ be a recursive system with the realization (17), and assume Σ is everywhere locally reachable with the reachability integer η. We denote by $R_\Sigma(x)$ the set of all states that are reachable from the state x in η steps, i.e.

$$R_\Sigma(x) := \operatorname{Im} f^\eta(x,\cdot)$$

Clearly, the realization is globally reachable if and only if $R_\Sigma(x) = R^n$ for all $x \in R^n$. The following statement indicates a large class of systems that are globally reachable.

Proposition 3: *Let Σ be a system with the recursive representation $x_{k+1} = f(x_k, u_k)$. Assume that Σ is everywhere locally reachable, and that $R_\Sigma(x)$ is a closed set for all x. Then, Σ is globally reachable.*

Proof: Assume that the realization (17) of Σ is everywhere locally reachable, and let η be its reachability integer. Let $M \subset R^n$ be a compact set, let $x \in M$ be a point, and let

$$R_\Sigma(x, M) := R_\Sigma(x) \cap M = \operatorname{Im} f^{\eta}(x, \cdot) \cap M$$

be the set of all points of M that are reachable from x in η steps. Assume that $R_\Sigma(x, M) \neq \varnothing$. Since $R_\Sigma(x)$ is a closed set and M is compact, it follows that $R_\Sigma(x, M)$ is a compact set.

Next, consider a point $x' \in R_\Sigma(x, M)$. There is then a list of input values $u_0, \ldots, u_{\eta-1} \in R^m$ such that $x' = f^{\eta}(x, u_0, \ldots, u_{\eta-1})$. The fact that (17) is everywhere locally reachable implies that there is a real number $\alpha(x') > 0$ for which the ball $B(\alpha(x'), x')$ in R^n (of radius $\alpha(x')$ and centre x') is contained in $\operatorname{Im} f^{\eta}(x, \cdot)$. Furthermore, we claim that there is a real number $\beta > 0$ so that one can take $\alpha(x') \geq \beta$ for all $x' \in R_\Sigma(x, M)$. Otherwise, there is a sequence of points $x_i \in R_\Sigma(x, M)$, $i = 1, 2, \ldots$, and a sequence of real numbers $\beta_i \to 0$ such that every ball $B(\alpha, x_i) \subset \operatorname{Im} f^{\eta}(x, \cdot)$ must have a radius $\alpha \leq \beta_i$. Since $R_\Sigma(x, M)$ is compact, the sequence $\{x_i\}$ has an accumulation point $x'' \in R_\Sigma(x, M)$. But, then, since $\beta_i \to 0$, the only ball centred at x'' and contained in $\operatorname{Im} f^{\eta}(x, \cdot)$ is the ball of zero radius, contradicting the fact that $f^{\eta}(x, \cdot)$ is an open function by local reachability.

We now show that every point of M is reachable from x in η steps. By contradiction, assume there is a point $z \in M$ that is not reachable from x in η steps. Then, in view of the previous paragraph, no point of the ball $B(\beta, z)$ of radius β around z is reachable from the state x in η steps. But then it follows by the same argument that every point of $B(\beta, z) \cap M$ must be contained in a ball of radius β all of whose points are not reachable from x in η steps; in other words, all points of the set $B(2\beta, z) \cap M$ are not reachable from x in η steps. Repeating this argument again and again, we obtain that for all integers $j \geq 1$, the points of the set $B(j\beta, z) \cap M$ are all not reachable from x in η steps. When the latter is combined with the fact that M (as a compact set) is bounded, it follows that no point of M is reachable from x in η steps. Thus, if M contains a point that is not reachable from x in η steps, then no point of M can be reachable from x in η steps. Or, equivalently, if M contains a point that is reachable from x in η steps, then all points of M must be reachable from x in η steps.

Now, there clearly is a state $x' \in R^n$ that is reachable from the state x. Let θ be a real number satisfying $\theta > |x'|$. Then, using the compact set $M := [-\theta, \theta]^n$, it follows from the conclusion of the last paragraph that the entire set $[-\theta, \theta]^n$ is reachable from the state x in η steps. Since this is true for every $\theta > |x'|$, it follows that every state $x'' \in R^n$ is reachable from the state x in η steps. Finally, since x is an arbitrary state, we conclude that the realization is globally reachable (in η steps). □

Using Propositions (2) and (3), we can easily determine in many cases of practical interest whether or not a system is locally and globally reachable. Here is a simple example that demonstrates the procedure.

Example 1: Consider the following system with the two dimensional state vector $x_k = (\xi_k, \zeta_k)^T$ and a single dimensional input u_k.

$$\begin{pmatrix} \xi_{k+1} \\ \zeta_{k+1} \end{pmatrix} = \begin{pmatrix} ((\xi_k)^2 + 1)\zeta_k \\ \xi_k\zeta_k + ((\xi_k)^2 + 1)u_k \end{pmatrix}$$

Here, we have $n = 2$, and the iterated recursion function becomes

$$f^2(\xi, \zeta, u_0, u_1) = \begin{pmatrix} \{[(\xi^2+1)\zeta]^2 + 1\}[\xi\zeta + (\xi^2+1)u_0] \\ (\xi^2+1)\zeta[\xi\zeta + (\xi^2+1)u_0] + \{[(\xi^2+1)\zeta]^2+1\}u_1 \end{pmatrix}$$

The matrix C of (18) is given by:

$$C(\xi, \zeta, u_0, u_1) = \begin{pmatrix} \{[(\xi^2+1)\zeta]^2+1\}(\xi^2+1) & 0 \\ (\xi^2+1)\zeta(\xi^2+1) & [(\xi^2+1)\zeta]^2+1 \end{pmatrix}$$

As one can see, we have $\det C(\xi, \zeta, u_0, u_1) \neq 0$ for all values of ξ, ζ, u_0, u_1, and whence the system is everywhere locally reachable. A slight reflection shows that the reachability integer in this case is $\eta = 2$.

Finally, direct observation shows that $\operatorname{Im} f^2(\xi, \zeta, \cdot) = R^2$ for all values of ξ, $\zeta \in R$, and whence the system is globally reachable. □

Remark 1: In many cases, one is interested in the behaviour of the system only over a bounded subset S of R^n. The present discussion can be directly adapted to such case by taking all notions relative to the subset S. □

By definition, a globally reachable system has the property that every state x' can be reached from every other state x within η steps. It is, of course, important to investigate the input lists that take the system from x to x'. The next statement shows that, for any compact subset M of R^n and for any states x, $x' \in M$, one can reach x' from x using an input list whose amplitudes do not exceed a bound that depends only on the set M, and not on the specific states x, x'.

Proposition 4: *Let Σ be a system having a realization of the form* (17) *with a continuous recursion function f. Assume the realization is everywhere locally reachable, as well as globally reachable, and let η be the reachability integer. Then, for every compact subset $M \subset R^n$, there is a real number $\mu \geq 0$ for which the following holds: for every pair of states $x, x' \in M$, there is an input list $u_0, \dots, u_{\eta-1} \in [-\mu, \mu]^m$ that takes the system from the state x to the state x'.*

Proof: Let (17) be a realization with a continuous recursion function that is everywhere locally reachable, as well as globally reachable, and let η be the reachability integer. Consider a compact subset $M \subset R^n$. Since the realization is globally reachable with reachability integer η, the following is true: for every pair of points x, $\xi \in M$, there is a list of input values $u_0(x, \xi), \dots, u_{\eta-1}(x, \xi) \in R^m$ such that $\xi = f^\eta(x, u_0(x, \xi), \dots, u_{\eta-1}(x, \xi))$. We have to show that there is a real number $\mu > 0$ so that one can choose this input list to satisfy $u_0(x, \xi), \dots, u_{\eta-1}(x, \xi) \in [-\mu, \mu]^m$ for all $x, \xi \in M$.

By contradiction, assume that there is no such real number $\mu > 0$. Then, there is a sequence of pairs of states $\xi_i, x_i \in M$, $i = 1, 2, 3, \ldots$ for which the following holds:

(*) Every list of input values $u_0(i), \ldots, u_{\eta-1}(i) \in R^m$ for which $\xi_i = f^{\eta}(x_i, u_0(i), \ldots, u_{\eta-1}(i))$ satisfies $|(u_0(i), \ldots, u_{\eta-1}(i))| \geq 2^i$, $i = 1, 2, \ldots$

By the compactness of M, the sequence of pairs $\{(\xi_i, x_i)\}_{i=1}^{\infty}$ has an accumulation point (ξ, x) in $M \times M$ i.e. there is a subsequence of the sequence $\{(\xi_i, x_i)\}_{i=1}^{\infty}$ that converges to the point $(\xi, x) \in M \times M$. For the sake of notational simplicity, assume that the entire sequence converges to this point, i.e. that $\text{limit}_{i\to\infty} (\xi_i, x_i) = (\xi, x)$. Since the realization is globally reachable, there is an input list $v_0, \ldots, v_{\eta-1} \in R^m$ such that $\xi = f^{\eta}(x, v_0, \ldots, v_{\eta-1})$.

Now, let $\varepsilon > 0$ be a real number, and denote by $B(\varepsilon, v_0, \ldots, v_{\eta-1}) \subset (R^m)^{\eta}$ the open ball of radius ε centred at the point $(v_0, \ldots, v_{\eta-1}) \in (R^m)^{\eta}$. Since $f^{\eta}(x, \cdot)$ is an open function, it follows that the image $f^{\eta}(x, B(\varepsilon, v_0, \ldots, v_{\eta-1}))$ contains a neighbourhood of the point ξ. There is then a real number $\alpha > 0$ such that the ball $B(\alpha, \xi) \subset R^n$ of radius α centred at ξ satisfies $B(\alpha, \xi) \subset f^{\eta}(x, B(\varepsilon, v_0, \ldots, v_{\eta-1}))$.

The continuity of the recursion function f implies that the iterated function f^{η} is continuous as well, and, since M is compact, f^{η} is uniformly continuous over M. Consequently, there is a real number $\beta > 0$ such that $|f^{\eta}(x, v_0, \ldots, v_{\eta-1}) - f^{\eta}(x', v_0, \ldots, v_{\eta-1})| < \alpha/2$ for all points $x, x' \in M$ satisfying $|x - x'| < \beta$.

Define the positive number $\gamma := \min\{\alpha/2, \beta\}$. Since $\text{limit}_{i\to\infty} (\xi_i, x_i) = (\xi, x)$, there is an integer $j \geq 1$ such that $|(\xi_i, x_i) - (\xi, x)| < \gamma$ for all $i \geq j$. Then, for every integer $i \geq j$ we have

$$|\xi_i - \xi| < \gamma \leq \alpha/2 \quad \text{and} \quad |x_i - x| < \gamma \leq \beta$$

and it follows that

$$|f^{\eta}(x_i, v_0, \ldots, v_{\eta-1}) - f^{\eta}(x, v_0, \ldots, v_{\eta-1})| < \alpha/2$$

Combining these facts, we obtain, for all $i \geq j$,

$$\begin{aligned}
|\xi_i - f^{\eta}(x_i, v_0, \ldots, v_{\eta-1})| &= |[\xi_i - \xi] - [f^{\eta}(x_i, v_0, \ldots, v_{\eta-1}) - \xi]| \\
&= |[\xi_i - \xi] - [f^{\eta}(x_i, v_0, \ldots, v_{\eta-1}) - f^{\eta}(x, v_0, \ldots, v_{\eta-1})]| \\
&\leq |[\xi_i - \xi]| + |f^{\eta}(x_i, v_0, \ldots, v_{\eta-1}) - f^{\eta}(x, v_0, \ldots, v_{\eta-1})| \\
&< \alpha/2 + \alpha/2 \\
&= \alpha
\end{aligned}$$

Thus, $\xi_i \in B(\alpha, f^{\eta}(x_i, v_0, \ldots, v_{\eta-1}))$ for all integers $i \geq j$. But then, by the definition of the radius α, this implies that $\xi_i \in f^{\eta}(x_i, B(\varepsilon, v_0, \ldots, v_{\eta-1}))$ for all $i \geq j$. This means that, for every $i \geq j$, the state ξ_i can be reached from the state x_i by using an input list that belongs to the ball $B(\varepsilon, v_0, \ldots, v_{\eta-1})$. The amplitude of this input list is clearly bounded by the number

$$[|(v_0, \ldots, v_{\eta-1})| + \varepsilon]$$

in contradiction to (*). This proves our assertion. □

We comment that in Proposition 4, the path from x to x' is not necessarily contained in the compact set M; only the starting point x and the end point x' are in M.

The next subsection deals with the connection between reachability and eigensets. We show that for reachable systems, eigensets can be calculated in a relatively simple manner. As eigensets form the foundation for the derivation of state feedback functions (see Theorem 1), the notion of reachability takes on a prominent role in the theory of robust control for nonlinear systems.

4.2. *Reachability, disturbances, and eigensets*

Let Σ be a system having the nominal recursive representation $x_{k+1} = f(x_k, u_k)$, $k = 0, 1, 2, \ldots$ Throughout the ensuing discussion we assume that the recursion function f of Σ is a continuous function, that Σ is everywhere locally reachable as well as globally reachable, and we let η be the reachability integer. The objective of this subsection is to present an effective computational technique for the calculation of eigensets of the recursion function f of Σ. As seen in Theorem 1, these eigensets can be used to construct state feedback functions that solve the design problem (7).

The fact that the system Σ is globally reachable implies that its state values can be assigned arbitrarily at steps that are integer multiples of the reachability integer η. In other words, for any sequence of vectors $\xi_0, \xi_1, \xi_2, \ldots \in R^n$, there is an input sequence u that drives Σ in such a way that the resulting trajectory x satisfies $x_0 = \xi_0$, $x_\eta = \xi_1$, $x_{2\eta} = \xi_2$, $x_{3\eta} = \xi_3, \ldots$, or $x_{i\eta} = \xi_i$ for all integers $i \geq 0$. Indeed, global reachability implies that for every integer $i = 0, 1, 2, \ldots$, there is an input list $u_0(i), \ldots, u_{\eta-1}(i) \in R^m$ such that

$$\xi_{i+1} = f^{\eta}(\xi_i, u_0(i), \ldots, u_{\eta-1}(i)) \tag{19}$$

The concatenated input sequence

$$u = u_0(0), \ldots, u_{\eta-1}(0),\ u_0(1), \ldots, u_{\eta-1}(1),\ u_0(2), \ldots, u_{\eta-1}(2), \ldots \tag{20}$$

clearly achieves the desired result. The availability of this input sequence is a basic tool for our ensuing discussion.

Note that although the states can be assigned arbitrarily at steps that are integer multiples of η, there is usually little choice when it comes to selecting the states through which the trajectory passes at steps that are not integer multiples of η. These states are restricted by system characteristics, and cannot be assigned arbitrarily. In order to satisfy the design problem (7), one has to guarantee, among others, that the amplitudes of these states do not exceed Δ. Whether or not this requirement can be satisfied depends on the value of Δ and on the characteristics of the recursion function f of Σ.

Consider now the problem of finding a state feedback function σ that satisfies (7) for some real numbers δ, $\Delta > 0$, where $\delta \leq \Delta$. According to Theorem 1, the existence of such a feedback function is guaranteed when the initial condition of the controlled system Σ is within the maximal (δ, Δ)-eigenset of the recursion function f of Σ.

Now, in order to be of practical significance, the set of permissible initial conditions of Σ should contain a ball around the origin, so that a range of initial conditions is admissible. This requirement is incorporated into design problem (7), and it requires the use of (δ, Δ)-eigensets that contain a ball $B(\rho, 0)$ of radius $\rho \geq 0$ around origin of R^n.

Note that for a given value of ρ, the actual set of possible initial conditions of Σ includes a ball of radius $(\rho + \delta)$, since an uncertainty of amplitude not exceeding δ is always permitted around each initial condition. Thus, the value $\rho = 0$ is a permissible selection here, and does not require absolute accuracy in the setting of the initial condition. Also, since all states used by the closed loop system must be of amplitude not exceeding Δ, we have the additional requirement that $\rho + \delta \leq \Delta$.

Assume now that the system Σ is started from an initial conditions $x_0 \in B(\rho, 0)$. In general, it is not possible to find an input sequence u that drives $\Sigma(x_0)$ so that all states along the resulting trajectory are within $B(\rho, 0)$, even in the case where no disturbances are present. In other words, as much as one would like to prevent further dispersion of the trajectory from the origin, it is usually impossible to maintain the entire trajectory within the ball of radius ρ, for every initial condition in that ball.

Now, when Σ is globally reachable with reachability integer η, and no disturbances are present, it follows by (19) and (20) that an input sequence u for Σ can be found for which the resulting nominal state sequence x satisfies

$$x_{k\eta} \in B(\rho, 0) \text{ for all integers } k \geq 0 \tag{21}$$

In other words, for this input sequence, the nominal trajectory re-enters the ball $B(\rho, 0)$ at least once every η steps.

Of course, since the values $x_{k\eta}$, $k = 1, 2, \ldots$, can be assigned arbitrarily (by choosing an appropriate input sequence u), one could restrict these values even further, and force them to be within a ball of radius smaller than ρ. In the present discussion, however, we use (21) as our guiding requirement. We also assume that ρ is a specified design parameter.

To summarize, we require the nominal trajectory x of the closed loop system (figure 1) to satisfy

$$|x_k| \leq \Delta - \delta \quad \text{and} \quad |x_{k\eta}| \leq \rho, \; k = 0, 1, 2, \ldots$$

where the first inequality takes into account the fact that a disturbance of amplitude not exceeding δ may be added to the nominal trajectory at each step.

In addition to the practical significance discussed so far, condition (21) also has important mathematical implications. As discussed below, this condition allows us to calculate a (δ, Δ)-eigenset of the recursion function f by examining only the first η steps of the recursion $x_{k+1} = f(x_k, u_k)$. This then yields a finite technique for the calculation of (δ, Δ)-eigensets.

To be somewhat more specific, but still assuming no disturbances are active, construct the following set of input lists. For every state $x \in B(\rho + \delta, 0)$, let $\mathcal{U}_\rho(x)$ be the set of all input lists $u_0, \ldots, u_{\eta-1} \in R^m$ for which the state $f^{\eta}(x, u_0, \ldots, u_{\eta-1})$ belongs to $B(\rho, 0)$. Of these input lists, let $\mathcal{U}(x)$ be the set of all input lists $u_0, \ldots, u_{\eta-1} \in \mathcal{U}_\rho(x)$ for which

$$|f^{i}(x, u_0(x), \ldots, u_{i-1}(x))| \leq \Delta - \delta, i = 1, \ldots, \eta$$

so that the state amplitude bound is not violated. Assume that $\mathcal{U}(x) \neq \emptyset$ for all $x \in B(\rho + \delta, 0)$ (otherwise, the design objective cannot be met).

For each list $u_0, \dots, u_{\eta-1} \in \mathscr{U}(x)$, construct the following set of (state, input) pairs through which the system passes

$$S(x, u_0, \dots, u_{\eta-1}) := \{(x, u_0), (f(x, u_0), u_1), \dots, (f^{\eta-1}(x, u_0, \dots, u_{\eta-2}), u_{\eta-1})\}$$

which is a subset of $R^n \times R^m$. Finally, combine all these (state input) pairs into one set

$$S := \bigcup_{x \in B(\rho,0), u_0,\dots,u_{\eta-1} \in \mathscr{U}(x)} S(x, u_0, \dots, u_{\eta-1})$$

which is again a subset of $R^n \times R^m$. In view of the fact that for all $x \in B(\rho + \delta, 0)$ we have $f^{\eta}(x, u_0, \dots, u_{\eta-1}) \in B(\rho, 0)$, it follows that $f[S] \subset \Pi_x S$, so that S is a conditional invariant subset of the recursion function f. Note that the calculation of S involves only consideration of η iterations of f. Every state obtained at step η is contained in the initial condition set $B(\rho + \delta, 0)$, and whence the previous input values can also be used for the steps η and beyond. Of course, this construction is facilitated by reachability. Incidentally, the set S also satisfies $B(\rho, 0) \subset \Pi_x S$ and $|f[S]| \le \Delta - \delta$.

The set S constructed above is not a (δ, Δ)-eigenset of the function f, since some effects of the disturbances have not been taken into account so far. Nevertheless, the technique used in the construction of S forms the basis of our derivation of a finite process for the calculation of (δ, Δ)-eigensets. We now proceed to develop this process.

Consider a system Σ having the recursive representation $x_{k+1} = f(x_k, u_k)$. For an initial condition $x \in R^n$, an integer $i \ge 0$, a list of input values $u_0, \dots, u_{i-1} \in R^m$, and a real number $\delta > 0$, we construct recursively a subset $f^i_\delta(x, u_0, \dots, u_{i-1}) \subset R^n$ as follows.

$$f^0_\delta := N_\delta(x)$$

$$f^k_\delta(x, u_0, \dots, u_{k-1}) := f\{N_\delta[f^{k-1}_\delta(x, u_0, \dots, u_{k-2})], N_\delta(u_{k-1})\}, \quad k = 1, \dots, i$$

In intuitive terms, the set $f^i_\delta(x, u_0, \dots, u_{i-1})$ consists of all states the system can reach at the step i under the following conditions: the system is started from the nominal initial condition x and is driven by the nominal input list $u_0, \dots, u_{i-1}$, while at each step the state value as well as the input value are disturbed by a disturbance of amplitude not exceeding δ.

Now, let $\Delta > 0$ be the specified bound on the disturbance effects, as in (7). Also, let δ, $\rho > 0$ be two real numbers satisfying $\rho + \delta \le \Delta$. For a state $x \in B(\rho, 0)$, consider the set of all input lists $u_0(x), \dots, u_{\eta-1}(x) \in R^m$ for which the following hold.

$$|f^i_\delta(x, u_0(x), \dots, u_{i-1}(x))| \le \Delta - \delta \quad \text{for all} \quad i = 1, \dots, \eta - 1 \tag{22}$$

and

$$|f^\eta_\delta(x, u_0(x), \dots, u_{\eta-1}(x))| \le \rho \tag{23}$$

Note that these relations simply represent a finite set of inequalities based on the recursion function f of Σ. Using the solution of these inequalities we can build a (δ, Δ)-eigenset of the recursion function f as follows.

First, for each state $x \in B(\rho, 0)$, let $U(x, \rho, \delta, \Delta) \subset (R^m)^\eta$ be the set of all input lists $(u_0(x), \ldots, u_{\eta-1}(x)) \in (R^m)^\eta$ that satisfy (22) and (23). In other words, $U(x, \rho, \delta, \Delta)$ is the solution set of the inequalities (22) and (23). Assume that $U(x, \rho, \delta, \Delta) \neq \varnothing$ for all $x \in B(\rho, 0)$; conditions on the recursion function f of Σ under which this assumption is valid are discussed later.

Next, for a list $(u_0(x), \ldots, u_{\eta-1}(x)) \in U(x, \rho, \delta, \Delta)$ and an integer $i \in 0, \ldots, \eta - 1$, denote by

$$\pi_i(u_0(x), \ldots, u_{\eta-1}(x)) := u_i(x)$$

the projection onto the ith element of the list. Build the following subsets of $R^n \times R^m$, consisting of state–input pairs.

$$S_0 := \{(y, u): y \in N_\delta(x), x \in B(\rho, 0), u \in \pi_0 U(x, \rho, \delta, \Delta)\}$$

$$S_i := \{(y, u): y \in f_\delta^i(x, u_0(x), \ldots, u_{i-1}(x)), x \in B(\rho, 0), u \in \pi_i U(x, \rho, \delta, \Delta)\}, \quad i = 1, \ldots, \eta - 1$$

Finally, combine these subsets into the set

$$S_\rho(\delta, \Delta) := \bigcup_{i=0,\ldots,\eta-1} S_i \tag{24}$$

which is a subset of $R^n \times R^m$. A slight reflection shows then that (22) and (23) imply the following.

Theorem 3: *Let Σ be a system with the recursive representation $x_{k+1} = f(x_k, u_k)$, and assume there are real numbers $\Delta > 0$, $\delta > 0$, and $\rho > 0$, where $\rho + \delta \leq \Delta$, for which the solution set $U(x, \rho, \delta, \Delta)$ of (22) and (23) is not empty for any $x \in B(\rho, 0)$. Then, the set $S_\rho(\delta, \Delta)$ of (24) is a (δ, Δ)-eigenset of the recursion function f, and $B(\rho, 0) \subset \Pi_x S_\rho(\delta, \Delta)$.*

In view of the fact that $S_\rho(\delta, \Delta)$ is obtained from the solution of the finite set of inequalities (22) and (23), we have a finite procedure for the calculation of (δ, Δ)-eigensets of the recursion function f. In general, $S_\rho(\delta, \Delta)$ is not equal to the maximal (δ, Δ)-eigenset of f. Nevertheless, the maximal (δ, Δ)-eigenset is not needed in order to construct a state feedback function according to Theorem 1. Using the procedure of Theorem 1, the eigenset $S_\rho(\delta, \Delta)$ allows us to build a state feedback function σ for which the closed loop system Σ_σ permits disturbances of amplitude not exceeding δ, and may be started from any initial condition of magnitude not exceeding ρ, all without violating the bound Δ on the output sequence.

Of course, when solving the inequalities (22) and (23), one has to obtain the largest possible value of δ for which a solution exists. For the largest value of δ, the closed loop system Σ_σ permits the largest disturbance amplitudes possible within the framework of the present section. In this way, we have obtained a computable solution for the design of a state feedback function σ.

As we can see, the critical step in this process is the solution of the set of inequalities (22) and (23). It is therefore important to examine conditions under which these inequalities possess a non-empty solution set. Before turning to this examination, we provide an example.

Example 2: Consider the system $\Sigma: S(R) \to S(R^2)$ with the following realization

$$x_{k+1} = \begin{pmatrix} 2x_{1,k} + [(x_{1,k})^2 + 1]u_k \\ x_{1,k} \end{pmatrix}$$

In this case,

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and the recursion function is

$$f(x_1, x_2, u) = \begin{pmatrix} 2x_1 + [(x_1)^2 + 1]u \\ x_1 \end{pmatrix}$$

Iterating the recursion function we obtain

$$f^2(x_1, x_2, u_0, u_1) = \begin{pmatrix} 4x_1 + 2[(x_1)^2 + 1]u_0 + \{[2x_1 + [(x_1)^2 + 1]u_0]^2 + 1\}u_1 \\ 2x_1 + [(x_1)^2 + 1]u_0 \end{pmatrix}$$

Take $\Delta = 1$ and $\rho = 1/2$. Then, we must have $\delta \leq \Delta - \rho = 1/2$, and (22) and (23) in this case lead to the following: For each $|x| \leq 1/2$, find u_0, u_1 for which the inequalities

$$|f(y_1, y_2, v_0)| \leq 1 - \delta \tag{a}$$

$$|f^2(y_1, y_2, v_0, v_1)| \leq 1/2 \tag{b}$$

are valid for all y_1, y_2, v_0, v_1 satisfying $|y_1 - x_1| \leq \delta$, $|y_2 - x_2| \leq \delta$, $|v_0 - u_0| \leq \delta$ and $|v_1 - u_1| \leq \delta$. It is usually easiest to solve these inequalities sequentially, by first solving the inequality (*a*), and then invoking (*b*) on the solution of (*a*). We adopt this technique below.

Consider first inequality (*a*). Since the second component of (*a*) is x_1, we need $1/2 + \delta \leq 1 - \delta$, which requires $\delta \leq 1/4$. For the first component, we assign the nominal input value

$$u_x = \frac{-2x_1}{(x_1)^2 + 1} \tag{c}$$

which leads to a nominal value of 0 for the first component. For a given nominal x_1 and disturbances δ_1 on x_1 and δ_2 on u, the value of the first component of (*a*) becomes (note that u_x is determined by the nominal value of x_1)

$$\left| 2(x_1 + \delta_1) = [(x_1 + \delta_1)^2 + 1]\left(\frac{-2x_1}{(x_1)^2 + 1} + \delta_2 \right) \right| =: a_1 \tag{d}$$

where $|x_1| \leq \rho = 1/2$, $|\delta_1| \leq \delta$, and $|\delta_2| \leq \delta$. To satisfy inequality (*a*), we need $a_1 \leq 1 - \delta$, which, since $\delta \leq 1/4$, requires $a_1 \leq 0.75$.

Now, due to the particular form of the recursion function f, the number a_1 is also the magnitude of the second component of f^2. In view of (*b*), we need therefore to require $a_1 \leq 1/2$. A numerical examination shows that the largest value of δ that satisfies the last requirement is approximately $\delta = 0.16$; for this value of δ, one has $a_1 \leq 0.49$ for all $|x_1| \leq 1/2$.

Consider next inequality (*b*), using the feedback assignment represented by (*c*). Namely, denoting by $f_1(x_1, x_2, v_0)$ the first component of $f(x_1, x_2, v_0)$, set

$$u_0 = \frac{-2x_1}{(x_1)^2 + 1}, \quad u_1 = \frac{-2f_1(x_1, x_2, u_0)}{[f_1(x_1, x_2, u_0)]^2 + 1}$$

From the preceding paragraph we have that $|f_1(x_1, x_2, u_0)| < 1/2$; thus, the argument used to show that inequality (a) holds for $\delta = 0.16$ implies that inequality (b) is also valid for $\delta = 0.16$. Consequently, $\delta = 0.16$ is a permissible value of δ in this case; as mentioned earlier, this is approximately the maximal permissible value of δ in this case. According to (c), the feedback function here is given by

$$\sigma(x_1, x_2) = \frac{-2x_1}{(x_1)^2 + 1} \qquad \square$$

We now turn to a discussion of some conditions under which the set of inequalities (22) and (23) is guaranteed to have a solution for some values of the real numbers Δ, δ, and ρ. The next statement shows that reachability guaranties the existence of a solution.

Theorem 4: *Let Σ be a system having the recursive representation $x_{k+1} = f(x_k, u_k)$ with a continuous recursion function f. Assume Σ is everywhere locally reachable as well as globally reachable. Then, for every real number $\rho > 0$, there are real numbers δ, $\Delta > 0$, where $\rho + \delta \leq \Delta$, for which the inequalities* (22) *and* (23) *have a non-empty solution set.*

Proof: Assume that the system Σ is everywhere locally reachable as well as globally reachable, and let η be its reachability integer. Let $\rho > 0$ be a real number, and consider the ball $B(2\rho, 0)$ in R^n. Since $B(2\rho, 0)$ is a compact set, it follows from Proposition 4 that there is a real number $\mu > 0$ that satisfies the following condition. For every point $x \in B(2\rho, 0)$, there is an input list $u_0(x), \ldots, u_{\eta-1}(x) \in [-\mu, \mu]^m$ such that $f^{\eta}(x, u_0(x), \ldots, u_{\eta-1}(x)) = 0$. Denote by $B^i(2\mu, 0)$ the ball of radius 2μ around the origin in the space $(R^m)^i, i = 1, \ldots, \eta$, and note that $B^i(2\mu, 0)$ is also a compact set.

The continuity of the function $f\colon R^n \times R^m \to R^n$ implies that all iterations $f^i\colon R^n \times (R^m)^i \to R^n$, $i = 1, \ldots, \eta$, are continuous functions. Consequently, compactness of the above mentioned sets implies that, for each $i = 1, \ldots, \eta$, there is a real number $N_i > 0$ such that

$$|f^i[B(2\rho, 0), B^i(2\mu, 0)]| \leq N_i, \quad i = 1, \ldots, \eta$$

Let $N := \max\{N_1, \ldots, N_\eta, \rho\}$, and take

$$\Delta := 2N \tag{25}$$

Consider now the restriction of the function f^i to the domain $B(2\rho, 0) \times B^i(2\mu, 0) \subset R^n \times (R^m)^i$, $i = 1, \ldots, \eta$. Since this domain is compact and f^i is a continuous function, it follows that f^i is uniformly continuous over $B(2\rho, 0) \times B^i(2\mu, 0)$. Consequently, for each $i = 1, \ldots, \eta - 1$, there is a real number $\alpha_i > 0$ for which the following holds for all $x \in B(2\rho, 0)$:

$$|f^i(x', v_0, \ldots, v_{i-1}) - f^i(x, u_0(x), \ldots, u_{i-1}(x))| < N_i$$

for all $x' \in B(2\rho, 0)$ satisfying $|x' - x| < \alpha_i$, and for all $v_0, \ldots, v_{i-1} \in [-2\mu, 2\mu]^m$ satisfying $|(v_0, \ldots, v_{i-1}) - (u_0(x), \ldots, u_{i-1}(x))| < \alpha_i$. Also, there is a real number $\alpha_\eta > 0$ such that

$$|f^{\eta}(x', v_0, \ldots, v_{\eta-1}) - f^{\eta}(x, u_0(x), \ldots, u_{\eta-1}(x))| < \rho$$

for all $x' \in B(2\rho, 0)$ satisfying $|x' - x| < \alpha_\eta$, and for all $v_0, \ldots, v_{\eta-1} \in [-2\mu, 2\mu]^m$ satisfying $|(v_0, \ldots, v_{\eta-1}) - (u_0(x), \ldots, u_{\eta-1}(x))| < \alpha_\eta$. Set

$$\delta := \min\{\alpha_1, \ldots, \alpha_\eta, \rho, \mu, N\} \tag{26}$$

It is then a direct consequence of (25) to (26) that (22) and (23) are satisfied for the present values of ρ, Δ, and δ. This concludes our proof. □

In particular, Theorem 4 shows that for a reachable system with a continuous recursion function, one can always find a robust state feedback controller that guarantees a bounded response over any prescribed range of initial conditions.

Generally speaking, the largest disturbance amplitude δ that can be permitted for a given system depends on the bound Δ imposed on the disturbance effects, on the characteristics of the recursion function f, and on the imposed initial condition radius ρ. For a specific recursion function f, there may be values of ρ and Δ for which the inequalities (22) and (23) have no solution. However, when the inequalities (22) and (23) have a solution for the desired values of Δ and ρ, the largest value of δ for which a solution exists can be calculated directly, as seen in Example 2. This value of δ determines the largest disturbance amplitude within the framework of the present section.

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