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FRACTION REPRESENTATIONS, NON-ADDITIVE STATE FEEDBACK, AND NONLINEAR CONTROL

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ABSTRACT

Two broad issues in the theory of nonlinear control are discussed - nonlinear static state feedback and the construction of coprime fraction representations. First, a theory of static state feedback, valid for a wide class of nonlinear systems, is developed. The theory yields explicit formulas for the computation of static state feedback functions that internally stabilize a nonlinear system. Next, it is shown that these stabilizing state feedback functions can be used to construct right coprime fraction representations for nonlinear systems, even in cases where the output of the system is not the state. The resulting coprime fraction representations have a particularly simple factorization space.

1. INTRODUCTION

The purpose of this note is to review the theory of nonlinear static state feedback developed in HAMMER [1989b], and to indicate its application to the construction of right coprime fraction representations of nonlinear systems, described in HAMMER [1989c]. Right coprime fraction representations are used to derive controllers that robustly stabilize and assign desirable dynamics to a nonlinear system (HAMMER [1988 and 1989a]). The presentation here is brief and qualitative; Proofs and more detailed discussions are provided in the references.

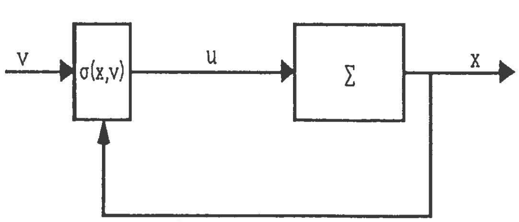
The theory of nonlinear state feedback of HAMMER [1989b] is valid for nonlinear discrete-time systems Σ of the form

(1.1) $\quad x_{k+1} = f(x_k, u_k),\ k = 0, 1, 2, \ldots .$

Here, $\{u_k\}_{k=0}^{\infty}$ is the input sequence of m-dimensional real vectors; $\{x_k\}_{k=0}^{\infty}$ is the output sequence of p-dimensional real vectors; and f, called the *recursion function*, is continuous. A system described by (1.1) is called an *input/state system*.

Let Σ be an input/state system, and consider the configuration

(1.2)



where the loop is closed through the continuous feedback function $\sigma(x,v)$. The closed loop has the input sequence $\{v_k\}_{k=0}^{\infty}$ of m-dimensional real vectors, and its input/output relation is denoted by Σ_σ. The recursive representation of Σ_σ is

(1.3) $\quad x_{k+1} = f(x_k, \sigma(x_k, v_k)),\ k = 0, 1, 2, \ldots .$

We explicitly derive the feedback functions σ for which the closed loop (1.2) is internally stable, whenever they exist.

Following HAMMER [1986], we restrict our attention to stabilization over bounded domains, by assuming that all possible input sequences v of the closed loop system (1.2) are of amplitude not exceeding a prespecified bound. This allows us to abide by realistic considerations relating to the maximal signal amplitudes permitted by the physical setup. It also yields a mathematical simplification of the stabilization problem.

Using the stabilizing state feedback functions σ, we then develop a method for the computation of right coprime fraction representations of nonlinear systems. These are representations of the form $\Sigma = PQ^{-1}$, where P and Q are stable and coprime nonlinear systems. The method is valid for systems Σ of the form

$$x_{k+1} = f(x_k, u_k),$$
(1.4)
$$y_k = h(x_k),\ k = 0, 1, 2, \ldots ,$$

where the input u is m-dimensional; the output y is p-dimensional; x is an intermediate q-dimensional 'state' variable; and f and h are continuous functions. A system Σ of the form (1.4) is said to have a *continuous realization*. The *input/state part* of Σ is the system Σ_s, induced by the recursion $x_{k+1} = f(x_k, u_k)$, $k = 0, 1, 2, \ldots$. To construct a right coprime fraction representation of Σ, suppose Σ_s is inserted into the closed loop (1.2), yielding the input/output relation $\Sigma_{s\sigma}$. Assume that σ was chosen so that $\Sigma_{s\sigma}$ is stable. Then, using σ, we construct in section 4 a right coprime fraction representation of the original system Σ. Once such fraction representation is known, the methods developed in HAMMER [1986, 1988a and 1988b] can be used to robustly stabilize Σ and assign desirable dynamics to the closed loop, without accessing the state x. In this way, σ is used only as a means to obtain a fraction representation, and the actual control configuration that stabilizes Σ requires no access to the state.

The present note is written within the framework of HAMMER [1984, 1986, 1987, 1989a, b, and c]. Recent alternative studies of the theory of fraction representations for nonlinear systems can be found in DESOER and LIN [1984], DESOER and KABULI [1988], TAY and MOORE [1988], SONTAG [1989a and b], KRENER [1989], the references listed in these papers, and others.

2. BASICS

Let $S(R^m)$ be the set of all sequences $(u_0, u_1, \ldots)$ of m-dimensional real vectors $u_i \in R^m$, $i = 0, 1, 2, \ldots$. Then, a system is simply a map $\Sigma : S(R^m) \to S(R^p)$, transforming m-dimensional input sequences into p-dimensional output sequences. The image of a subset $S \subset S(R^m)$ through Σ is denoted by $\Sigma[S]$, and consists of all elements $x \in S(R^p)$ satisfying $x = \Sigma u$ for some element $u \in S$.

Let $|u| := \max\{|u^i|, i = 1, ..., m\}$ be the maximal absolute value of the coordinates of a vector $u \in R^m$. For a sequence $u \in S(R^m)$, denote $|u| := \sup_{i\geq 0} |u_i|$, so that $|\cdot|$ is the usual ℓ^∞-norm. We shall also employ a weighted ℓ^∞-norm ρ, given by $\rho(u) := \sup_{i\geq 0} 2^{-i}|u_i|$ for all $u \in S(R^m)$. For a set $S \subset S(R^m)$, we denote by $\bar{S}$ its closure with respect to ρ. Unless stated otherwise, continuity of maps is with respect to ρ.

To consider bounded spaces, denote by $S(\theta^m)$, where $\theta > 0$, the set of all sequences $u \in S(R^m)$ satisfying $|u| \leq \theta$. A system $\Sigma : S(R^m) \to S(R^p)$ is said to be *BIBO* (Bounded-Input Bounded-Output)-*stable* if, for every real number $\theta > 0$, there is a real number $M > 0$ such that $\Sigma[S(\theta^m)] \subset S(M^p)$. A system $\Sigma : S(R^m) \to S(R^p)$ is *stable* if the following hold: (i) Σ is BIBO-stable, and (ii) for every real number $\theta > 0$, the restriction $\Sigma : S(\theta^m) \to S(R^p)$ is continuous (with respect to ρ) (HAMMER [1984, 1986, 1988]). This notion of stability conforms with Lyapunov theory. Given to subsets $S_1 \subset S(R^m)$ and $S_2 \subset S(R^p)$ and a system $M : S_1 \to S_2$, we say that M is *unimodular* if it has a set theoretic inverse M^{-1}, and if M and M^{-1} are both stable.

Critical to our discussion is the following class of systems (HAMMER [1985, 1987]).

(2.1) DEFINITION. $\Sigma : S(R^m) \to S(R^p)$ is a *homogeneous system* if, for every real number $\alpha > 0$ and for every subset $S \subset S(\alpha^m)$, the following holds true. Whenever there exists a real number $\theta > 0$ such that $\Sigma[S] \subset S(\theta^p)$, the restriction of Σ to the closure $\bar{S}$ of the set S in $S(\alpha^m)$ is a continuous map $\Sigma : \bar{S} \to S(\theta^p)$.□

In qualitative terms, a system is homogeneous if it is continuous whenever its outputs are bounded. Thus, a homogeneous system is stable (i.e., bounded and continuous) whenever it is BIBO-stable. In view of the following statement (HAMMER [1987]), all the systems considered here are homogeneous.

(2.2) PROPOSITION. A system $\Sigma : S(R^m) \to S(R^p)$ having a continuous realization is a homogeneous system.

We turn now to coprimeness and coprime fraction representations (HAMMER [1985, 1987]). ($P^*[S]$ denotes the set of all input sequences u for which $Pu \in S$, namely, the inverse image.)

(2.3) DEFINITION. Let $S \subset S(R^q)$ be a subset. Two stable systems $P : S \to S(R^p)$ and $Q : S \to S(R^m)$ are *right coprime* if the following conditions hold.

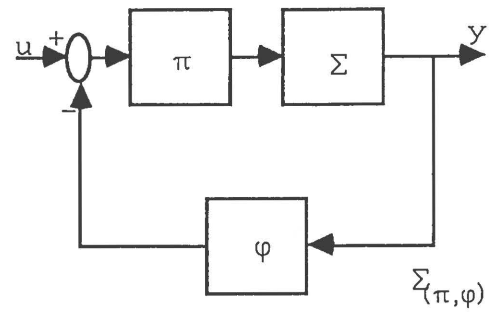
(i) For every real number $\tau > 0$ there exists a real number $\theta > 0$ such that

$$P^*[S(\tau^p)] \cap Q^*[S(\tau^m)] \subset S(\theta^q).$$

(ii) For every real number $\tau > 0$, the set $S \cap S(\tau^q)$ is a closed subset of $S(\tau^q)$ (with respect to ρ).□

Intuitively, P and Q are right coprime if, for every unbounded input sequence u, at least one of the output sequences Pu or Qu is unbounded. A *right coprime fraction representation* of a system $\Sigma : S(R^m) \to S(R^p)$ is of the form $\Sigma = PQ^{-1}$, where $P : S \to S(R^p)$ and $Q : S \to S(R^m)$ are stable right coprime systems, with Q being a set isomorphism. The space S, which is required to be contained in $S(R^q)$ for some integer $q > 0$, is called the *factorization space* . To indicate the significance of right coprime fraction representations, consider the following configuration.

(2.4)



Here, $\Sigma : S(\alpha^m) \to S(R^p)$ is the given system that needs to be controlled (α describes the largest input amplitude the system Σ permits); $\pi : S(R^m) \to S(R^m)$ is a dynamic precompensator; and $\varphi : S(R^p) \to S(R^m)$ is a dynamic feedback compensator, connected additively. The closed loop system is denoted by $\Sigma_{(\pi,\varphi)}$. It is particularly convenient to choose (HAMMER [1985])

$$(2.5) \qquad \pi = B^{-1}, \quad \varphi = A,$$

where $A : S(R^p) \to S(R^m)$ and $B : S(R^m) \to S(R^m)$ are stable systems, with B being a set isomorphism. Of course, A and B^{-1} have to be causal. Now, Let $\Sigma = PQ^{-1}$ be a right coprime fraction representation, and let $S \subset S(R^q)$ be its factorization space. Then (HAMMER [1986]),

$$(2.6) \qquad \Sigma_{(\pi,\varphi)} = P[AP + BQ]^{-1}.$$

Now, if the stable systems A and B are chosen so that

$$(2.7) \qquad AP + BQ = M,$$

where M is a *unimodular* system, we get

$$(2.8) \qquad \Sigma_{(\pi,\varphi)} = PM^{-1},$$

and the closed loop is input/output stable. In fact, the closed loop system will be internally stable under these circumstances, if the systems A and B satisfy some additional mild conditions (HAMMER [1986]). Let S' be the input domain of M. For consistency of (2.7) and (2.8), S' must be contained in the domain of P and Q, namely, in the factorization space S, so

$$(2.9) \qquad S' \subset S.$$

The following aspect of (2.8) is of particular interest. In general, the space of input sequences of the closed loop $\Sigma_{(\pi,\varphi)}$ is of the form $S(\theta^m)$, where $\theta > 0$ describes the maximal input amplitude permitted by the physical setup. Then, by (2.8), the domain of M^{-1}, which is the codomain of M, must be $S(\theta^m)$, so that $M : S' \to S(\theta^m)$, and S' is homeomorphic to $S(\theta^m)$ (HAMMER [1986]). Now, for stabilization, we need to construct M. A substantial simplification results when the factorization space S is itself of the form $S(\beta^m)$ for some real $\beta > 0$, since the construction of homeomorphisms $M : S(\beta^m) \to S(\theta^m)$ is a straightforward task. The basic advantage of the right coprime fraction represetnations constructed in this note is that they all have factorization spaces of the from $S(\beta^m)$ for some real $\beta > 0$, and thus are particularly adequate for use in stablization.

3. FEEDBACK FUNCTIONS, STABILIZATION, AND EIGENSETS

We review now the theory of stabilization by static state feedback of HAMMER [1989b]. Two fundamental properties of feedback are critical to us - continuity and reversibility. Continuity is quite obvious - we require the feedback function σ to be continuous. Reversibility requires the feedback operation to be reversible in the sense that it can be 'undone' by another feedback operation. Specifically, let $\sigma : R^p \times R^m \to R^m$, $(x,v) \mapsto \sigma(x,v)$, be a function. For every element $x \in R^p$, denote by $\sigma_x : R^m \to R^m$ the partial function given by $\sigma_x(v) := \sigma(x,v)$, $v \in R^m$. When Σ is given by (1.1), the recursion function f_σ of Σ_σ is

(3.1) $f_\sigma(x,v) = f(x,\sigma_x(v))$.

Assume next that the system Σ_σ is itself enclosed in a feedback loop, using the feedback function $\omega : R^p \times R^m \to R^m$, $(x,w) \mapsto \omega(x,w) = v$, so that there are now two feedback loops around Σ, and let $\Sigma_{\sigma\omega}$ denote the final system. As before, the recursion function $f_{\sigma\omega}$ of $\Sigma_{\sigma\omega}$ is

(3.2) $f_{\sigma\omega}(x,w) = f_\sigma(x,\omega_x(w)) = f(x,\sigma_x\omega_x(w))$.

Now, require that ω 'undoes' the feedback operation induced σ, so that $f_{\sigma\omega}(x,w) = f(x,w)$ for all x and w. For the latter to hold for any f, we need $w = \sigma_x\omega_x(w)$, and ω_x must be a right inverse of σ_x for every x. This implies that σ_x must be surjective (onto) for all x. The function ω_x, being a right inverse of σ_x, must then be injective (one to one). But then, requirng that both σ and ω induce reversible feedback operations, it follows that σ_x and ω_x must both be injective and surjective, i.e., set isomorphisms. Clearly, $\sigma_x : R^m \to \mathrm{Im}\,\sigma_x$ is a set isomorphism exactly when $\sigma_x : R^m \to R^m$ is injective. This leads to the following (HAMMER [1989b])

(3.3) DEFINITION. Let $\Sigma : S(R^m) \to S(R^p)$ be an input/state system. A *reversible feedback function* for Σ is a continuous function $\sigma : R^p \times R^m \to R^m$ for which the partial function $\sigma_x : R^m \to R^m$ is injective for any state x.□

We turn now to stability. The configuration (1.2) is *input/output stable* (for input sequences bounded by $\theta > 0$) if the restriction $\Sigma_\sigma : S(\theta^m) \to S(R^p)$ is stable. As is well known, the notion of input/output stability is too weak for practical applications. The feedback configurations discussed here are all internally stable in the sense that small noises added to the outputs of Σ or σ donot destroy stability. To incorporate noise effects, let Σ be given by the recursion

$$(3.4)\quad \begin{aligned} x_{k+1} &= f(x_k,u_k) + n_{k+1},\ k = 0, 1, 2, \dots, \\ x_0 &= x_{00} + n_0, \end{aligned}$$

where $n \in S(R^p)$ is a noise sequence, and where x_{00} is the specified nominal initial condition. Similarly, the output of the feedback is given by

(3.5) $u_k = \sigma(x_k,v_k) + \nu_k$, $k = 0, 1, 2, \dots$,

where $\nu \in S(R^m)$ is a noise sequence, and $v \in S(R^m)$ is the input sequence of the closed loop. Denote by $\Sigma_{\sigma,n,\nu}$ the input/output relation of the closed loop system with the noises n and ν present. Clearly, the system $\Sigma_{\sigma,n,\nu}$ can be regarded as a system accepting the three input sequences v, n, and ν, so write $\Sigma_{\sigma,n,\nu} : S(R^m)\times S(R^p)\times S(R^m) \to S(R^p)$, where the terms of the cross product correspond to v, n, and ν, respectively. The noises n and ν are assumed to have 'small' amplitudes, not exceeding a bound denoted by ε.

(3.6) DEFINITION. The configuration (1.2) is *internally stable* (for input sequences bounded by θ) if there is a real number $\varepsilon > 0$ such that $\Sigma_{\sigma,n,\nu} : S(\theta^m)\times S(\varepsilon^p)\times S(\varepsilon^m) \to S(R^p)$ is a stable system.□

One of the advantages of our setup is the simplicity it yields in the treatment of the notion of stability. Specifically, systems that possess a continuous realization are homogeneous by Proposotopn (2.2), and whence, by Definition (2.1), are stable whenever they are BIBO-stable. We emphasize that stability *includes* continuity. From this fact, the following follows (HAMMER [1989b]).

(3.7) PROPOSITION. Let $\Sigma : S(R^m) \to S(R^p)$ be a system having a recursive reresentation of the form $x_{k+1} = f(x_k,u_k)$, where $f : R^p \times R^m \to R^p$ is a continuous function. Let $\sigma : R^m \times R^p \to R^m$ be a reversible feedback function, and let $\theta > 0$ be a real number. Then, the system Σ_σ is internally stable (for input sequences bounded by θ) if and only if there is a pair of real numbers $\varepsilon, \delta > 0$ such that $\Sigma_{\sigma,n,\nu}[S(\theta^m)\times S(\varepsilon^p)\times S(\varepsilon^m)] \subset S(\delta^p)$.

In order to define a basic notion of our state feedback theory, we need some notation. Let $\varepsilon > 0$ be a real number, and let $S \subset R^n$ be a set. Denote by $B_\varepsilon(S)$ the open neighbourhood of S consisting of all points $y \in R^n$ for which there is a point $x \in S$ such that $|y - x| < \varepsilon$. Also, let $\Pi_p : R^p \times R^m \to R^p$ be the standard projection onto the first p coordinates.

(3.8) DEFINITION. An *eigenset* E of a function $f : R^p \times R^m \to R^p$ is a subset $E \subset R^p \times R^m$ satisfying $f[E] \subset \Pi_p[E]$. An *ε-eigenset* $\mathcal{E}$ of the function f is a subset $\mathcal{E} \subset R^p \times R^m$ satisfying the condition $f[B_\varepsilon(\mathcal{E})] \subset \Pi_p[\mathcal{E}]$, where $\varepsilon > 0$ is a real number.□

The next important notion of our stabilization theory is the notion of a uniform graph. Recall that the graph of a function $g : R^p \to R^m$ is simply a subset of $R^p \times R^m$ consisting of all points of the form $(x,g(x))$, $x \in R^p$. For a set $S \subset R^p \times R^m$ and a point $x \in R^p$, let $S(x)$ be the set of all $y \in R^m$ for which $(x,y) \in S$. A *uniform graph* is a subset $S \subset R^p \times R^m$ for which there is a continuous function $g : R^p \to R^m$ and a real number $\varsigma > 0$ such that $\mathcal{B}_\varsigma(g(x)) \subset S(x)$ for all $x \in \Pi_p[S]$. The function g is then called a *graphing function* for the set S. The notion of a uniform graph is quite simple on an intuitive level. First, a uniform graph S contains the graph of the continuous function g. Furthermore, it also contains the graph of any continuous function g' which differes from g by less than ς, namely, any continuous function g' saitsfying $|g'(x) - g(x)| < \varsigma$ for all $x \in \Pi_p[S]$. The notion of a uniform graph is a natural tool for the description of functions whose values may be corrupted by noise.

(3.9) DEFINITION. A continuous function $f : R^p \times R^m \to R^p$ is *uniformly conductive* at a point $x_0 \in R^p$ if it has a bounded ε-eigenset $\mathcal{E}$ for which the set $\mathcal{B}_\varepsilon(\mathcal{E})$ is a uniform graph, and $x_0 \in \Pi_p[\mathcal{E}]$.□

We can now state our main result.

(3.10) THEOREM. Let $\Sigma : S(R^m) \to S(R^p)$ be a system having a recursive representation $x_{k+1} = f(x_k,u_k)$ with the initial condition x_{00}, where $f : R^p \times R^m \to R^p$ is a continuous function. Let $\theta > 0$ be a real number. Then, the following two statements are equivalent.

(i) There exists a reversible state feedback function $\sigma : R^p \times R^m \to R^m$ for which the closed loop system $\Sigma_\sigma : S(\theta^m) \to S(R^p)$ is internally stable.

(ii) The recursion function f is uniformly conductive at the point x_{00}.

Thus, we have a complete characterization of internal stabilizability by static state feedback. The significance of this result is twofold. First, from a theoretical point of view, it provides a direct link between properties of the given recursion function f of the system that needs to be stabilized, and the existence of a stabilizing state feedback. From a practical point of view, eigensets of functions can be quite readily computed (HAMMER [1989b]). Once the eigensets are known, one can check whether f is uniformly conductive, and, if it is, stabilizing feedback functions σ for the system Σ can be directly derived (HAMMER [1989b]). This yields then an explicit procedure for the computation of stabilizing feedback functions σ.

The construction of the stabilizing feedback functions proceeds as follows. Let $\mathcal{E}$ be a bounded ε-eigenset over which the recursion function f of the system Σ is conductive. Then, there is a reversible feedback function $\sigma : R^p \times R^m \to R^m$ satisfying the following condition for some real number $0 < \varsigma < \varepsilon$ and for all $x \in \Pi_p[\mathcal{B}_\varepsilon(\mathcal{E})]$.

(3.11) $\sigma_x[-\theta,\theta]^m \subset \mathcal{B}_\varsigma(\mathcal{E})(x)$.

It can be shown (HAMMER [1989b]) that the feedback function σ internally stabilizes the system Σ, as follows.

(3.12) THEOREM. Let $\Sigma : S(R^m) \to S(R^p)$ be a system having a recursive representation $x_{k+1} = f(x_k,u_k)$ with the initial condition x_{00}, where $f : R^p \times R^m \to R^p$ is a continuous function, and let $\theta > 0$ be a real number. Assume that the recursion function f is uniformly conductive at the point x_{00}, and let $\mathcal{E}$ be an ε-eigenset of f for which $\mathcal{B}_\varepsilon(\mathcal{E})$ is a uniform graph and $x_{00} \in \Pi_p[\mathcal{E}]$. Then, every reversible feedback function $\sigma : R^p \times R^m \to R^m$ satisfying (3.11) yields an internally stable closed loop system $\Sigma_\sigma : S(\theta^m) \to S(R^p)$.

Generally speaking, in order to find a feedback function σ that satisfies (3.11), one has to construct a continuous family $\{\sigma_x\}$ of homeomorphisms $\sigma_x : [-\theta,\theta]^m \to \mathrm{Im}\,\sigma_x$, for which $\mathrm{Im}\,\sigma_x \subset \mathcal{B}_\varsigma(\mathcal{E})(x)$ for all $x \in \Pi_p[\mathcal{B}_\varepsilon(\mathcal{E})]$. Quite usually, the construction of all possible families $\{\sigma_x\}$ is not an easy problem, and, as well known, it is the subject of homotopy theory. However, some of the families $\{\sigma_x\}$ are quite easy to construct; One such family $\{\sigma_x\}$, i.e., one reversible feedback function σ, is the following (HAMMER [1989b]).

(3.13) COROLLARY. In the notation of Theorem (3.12), let g be a graphing function for $\mathcal{B}_\varepsilon(\mathcal{E})$. Let $r : R^m \to R$ be a continuous scalar positive valued function satisfying the following conditions: (i) There is a real number $\kappa > 0$ such that $r(x) \geq \kappa$ for all $x \in R^m$, and (ii) There is a real number $0 < \varsigma < \varepsilon$ such that, for every $x \in \Pi_p[\mathcal{B}_\varepsilon(\mathcal{E})]$, the ball $\mathcal{B}_{r(x)}(g(x)) \subset \mathcal{B}_\varsigma[\mathcal{E}](x)$. Define the function

$$\sigma(x,v) := [r(x)/\theta]v + g(x).$$

Then, σ is a reversible feedback function, and the closed loop system $\Sigma_\sigma : S(\theta^m) \to S(R^p)$ is internally stable.

The stabilizing feedback functions of the Corollary can be readily constructed in practice. Finally, when the given system Σ is a linear finite-dimensional time-invariant system, the class of feedback functions described by Corollary (3.13) includes the classical linear state feedback functions (HAMMER [1989b]).

4. THE CONSTRUCTION OF EIGENSETS AND FEEDBACK FUNCTIONS

The process of computing a static state feedback function that stabilizes a given nonlinear input/state system can be divided into the following three main steps. Let Σ be the system that needs to be stabilized, and let f be its recursion function.

1) Find an appropriate ε-eigenset $\mathcal{E}$ of f.

2) Find a graphing function for $\mathcal{E}$.

3) Find a stabilizing state feedback function σ, using Theorem (3.12) or Corollary (3.13).

Of course, all this is under the assumption that the function f is uniformly conductive; otherwise, by Theorem (3.10), stabilization is impossible.

The computation of ε-eigensets of functions involves the solution of certain sets of inequalities. More specifically, let $\Sigma : S(R^m) \to S(R^p)$ be an input/state system with the recursive representation $x_{k+1} = f(x_k,u_k)$, where $f : R^p \times R^m \to R^p$ is a continuous function. Assume that the system Σ needs to be stabilized over a range of output amplitudes $|x| \leq \delta$. Then, we need to find an ε-eigenset $\mathcal{E}$ of the function f for which the projection $\Pi_p[\mathcal{E}]$ onto the state space is bounded by δ. This can be handled in the following way. Find a subset $\mathcal{X} \subset R^p$ and a real number $\varsigma > 0$ for which the subsequent conditions hold. (i) $\mathcal{X} \subset [-\delta,\delta]^p$; (ii) For each element $x \in \mathcal{B}_\varsigma(\mathcal{X})$ there is a nonempty bounded subset $\mathcal{U}(x) \subset R^m$ such that $f[\mathcal{B}_\varsigma(x),\mathcal{B}_\varsigma(\mathcal{U}(x))] \subset \mathcal{X}$; and (iii) There is a real number $\alpha > 0$ such that $\mathcal{U}(x) \subset [-\alpha,\alpha]^m$ for all $x \in \mathcal{X}$. Then, it follows directly from the definitions that the set

(4.1) $\mathcal{E} := \{(x,u) \in R^p \times R^m : x \in \mathcal{X} \text{ and } u \in \mathcal{U}(x)\}$,

is an ε-eigenset of the function f for $\varepsilon = \varsigma$. This procedure will, in general, yield a class $\mathbb{E}$ of ε-eigensets of the function f, where $\mathbb{E}$ is empty in case no such ε-eigensets exist. Now, there are two possibilities - either $\mathbb{E}$ contains an ε-eigenset $\mathcal{E}$ for which $\mathcal{B}_\varepsilon(\mathcal{E})$ is a uniform graph, or it does not. In the first case, let $\mathcal{E}_g \in \mathbb{E}$ be an ε-eigenset for which $\mathcal{B}_\varepsilon(\mathcal{E}_g)$ is a uniform graph, and let $g(x)$ be a graphing function for $\mathcal{B}_\varepsilon(\mathcal{E}_g)$. Then, a stabilizing feedback function σ for the system Σ can be directly computed using Theorem (3.12) or Corollary (3.13). Otherwise, if $\mathbb{E}$ does not contain an ε-eigenset $\mathcal{E}$ for which $\mathcal{B}_\varepsilon(\mathcal{E})$ is a uniform graph, it follows by Theorem (3.10) that the system Σ cannot be internally stabilized with output amplitude bounded by the specified bound δ. However, it may still be possible to internally stabilize the system Σ if the output amplitude bound δ is increased.

In qualitative terms, condition (ii) of the previous paragraph is a controllability type condition. It requires that, for every state $x \in \mathcal{X}$, there be a set $\mathcal{U}(x)$ of input values u that steer the state so that it stays within the set $\mathcal{X}$, even if errors (of amplitudes not exceeding ς) in x or in u are present. Condition (iii) simply requires all relevant input values to have bounded amplitudes, and is usually just a formality.

EXAMPLE. Consider the system $\Sigma : S(R) \to S(R^2)$ described by the recursion

$$\begin{pmatrix} x_{1,k+1} \\ x_{2,k+1} \end{pmatrix} = \begin{pmatrix} (x_{1,k})^2 + x_{2,k} \\ \sin x_{2,k} + (1+(x_{2,k})^2)u_k \end{pmatrix},$$

where x_1, x_2 are the coordinates of the state vector, and where the nominal initial condition is $x_{00} = 0$. The recursion function is then

$$f(x,u) = \begin{pmatrix} (x_1)^2+x_2 \\ \sin x_2+(1+(x_2)^2)u \end{pmatrix}.$$

The output sequences of the closed loop system (including the noise) are required to be bounded by the real number $\delta > 0$; the input sequences are taken from $S(\theta)$, where $\theta > 0$ is a specified real number.

Using the methods described in the previos section, a stabilizing and reversible state feedback function for this system is computed in HAMMER [1989b]. The resulting function is given by

$$\sigma(x,v) := \lambda(x)v + g(x),$$

where

$$g(x) = \frac{(\eta_2-\eta_1)/2-[(x_1)^2+x_2]^2-\sin x_2}{1+(x_2)^2},$$

and

$$\lambda(x) = \frac{(\eta_1+\eta_2)}{2[1+(x_2)^2]\theta}.$$

Here, η_1 and η_2 are constants, which depend on δ and θ. The feedback function σ internally stabilizes the system Σ, subject to the above requirements.

As we have seen throughout our discussion, and, in particular in Theorem (3.10), the notion of a uniformly conductive function is the most fundamental notion of the theory of static state feedback for nonlinear systems. An input/state system is internally stabilizable if and only if its recursion function is uniformly conductive. Some further simple and explicit characterizations of uniformly conductive functions are provided in HAMMER [1989b].

5. FRACTION REPRESENTATIONS

In the present section we discuss the construction of right coprime fraction representations $\Sigma = PQ^{-1}$ whose factorization space is of the form $S(\beta^m)$, $\beta > 0$, following HAMMER [1989c]. The construction is based on the theory of static state feedback reviewed in the previous sections, and the results apply to systems Σ possessiong a continuous realization, for which the input/state part Σ_s is stabilizable. We consider first the case of input/state systems.

Let Σ be an input/state system, and assume there is a reversible feedback function $\sigma : R^p \times R^m \to R^m$ for which the system $\Sigma_\sigma : S(\theta^m) \to S(R^p)$ of (1.2) is stable. The existence of σ was discussed in the previous section. Now, referring to (1.2), let $v \in S(\theta^m)$ be an input sequence of the closed loop system, and let $u \in S(R^m)$ be the correponding input sequence of the system Σ, so that

(5.1) $\quad u = \sigma(\Sigma u,v),$

by which we simply mean that $u_k = \sigma(\Sigma u]_k,v_k)$ for all integers $k \geq 0$, where $\Sigma u]_k$ is the k-th element of the output sequence Σu. In view of the definition of a reversible feedback function, the partial function $\sigma_x : [-\theta,\theta]^m \to \sigma_x[-\theta,\theta]^m$ is a set isomorphism for every state x, and thus has an inverse function $\sigma_x^{-1} : \sigma_x[-\theta,\theta]^m \to [-\theta,\theta]^m$. Denote $\sigma^*(x,u) := \sigma_x^{-1}(u)$. Let S_u denote the set of all sequences $u \in S(R^m)$ that appear as input sequences of the system Σ in the closed loop (1.2) when v varies over $S(\theta^m)$, namely,

(5.2) $\quad S_u := \{u \in S(R^m) : u = \sigma(\Sigma u,v),\ v \in S(\theta^m)\}.$

Then we can write

(5.3) $\quad v = \sigma^*(\Sigma u,u)$

for all sequences $u \in S_u$. Let $\ell : S_u \to S(\theta^m)$ be the system given by

(5.4) $\quad \ell(u) := \sigma^*(\Sigma u,u),$

so that $v = \ell(u)$. In this notation, $\Sigma u = \Sigma_\sigma \ell u$ for all $u \in S_u$, and it follows that the restriction $\Sigma : S_u \to S(R^p)$ satisfies $\Sigma = \Sigma_\sigma \ell$. Now, assume for a moment that ℓ has a stable inverse $\ell^{-1} : S(\theta^m) \to S_u$, and denote

(5.5) $\quad Q := \ell^{-1} : S(\theta^m) \to S_u.$

Then, recalling that Σ_σ is stable, and setting

(5.6) $\quad P := \Sigma_\sigma : S(\theta^m) \to S(R^p),$

we obtain the right fraction representation

(5.7) $\quad \Sigma = PQ^{-1},$

which is valid over the input space S_u, and which has the factorization space $S(\theta^m)$. As it turns out, this fraction representation is in fact coprime, and thus a coprime fraction representation having the factorization space $S(\theta^m)$ is obtained, in line with our basic objective. In HAMMER [1989c] we proved the following statement, which shows that ℓ is in fact bicausal, and thus invertible. Recall that a system is *bicausal* if it is invertible, and if it and its inverse are both causal systems.

(5.8) LEMMA. The system $\ell : S_u \to S(\theta^m)$ of (5.4) is a bicausal isomorphism.

The following statement guaranties the stability of Q in the fraction representation (5.7) (HAMMER [1989c]).

(5.9) LEMMA. The system $Q : S(\theta^m) \to S_u$ is a stable system.

Moreover, an explicit representation of Q was derived in the above reference, and it is

(5.10)
$$\begin{pmatrix} w_{k+1} \\ x_{k+1} \end{pmatrix} = \begin{pmatrix} \sigma(f(x_k,w_k),v_k) \\ f(x_k,w_k) \end{pmatrix},\quad \begin{pmatrix} w_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} \sigma(x_0,v_0) \\ x_0 \end{pmatrix}$$
$$u_k = w_k,$$

$k = 0, 1, 2, \ldots$, where $v \in S(\theta^m)$ is the input sequence of Q and $u = Qv$ is the output sequence of Q.

An uttermost important property of the fraction representation (5.7) is the following (HAMMER [1989c]).

(5.11) LEMMA. The systems $P : S(\theta^m) \to S(R^p)$ and $Q : S(\theta^m) \to S_u$ of (5.7) are right coprime.

Thus, using the stabilizing reversible feedback function σ, we have constructed a right coprime fraction representation of the system Σ. The main advantage of this fraction representation over the fraction representations derived in HAMMER [1987] is that the current fraction representation has the factorization space $S(\theta^m)$. As discussed earlier in this note, the latter is highly instrumental in the construction of compensators that yield stabilization and desired dynamics assignment for the closed loop system (2.4) (HAMMER [1988 and 1989a]). It is also important to note that the numerator P and the denominator Q of our fraction representation are both implementable systems ((5.10), (5.6), (1.3)), and that Q is bicausal (Lemma (5.8)).

We turn now to the construction of right coprime fraction representations for systems possessing continuous realizations, using the results discussed so far in this section. Consider a nonlinear system $\Sigma : S(R^m) \to S(R^p)$ having the continuous realization (1.1), where $f : R^q \times R^m \to R^q$ and $h : R^q \to R^p$ are continuous functions, and where $u \in S(R^m)$ is the input sequence of Σ, $y = \Sigma u$ is the output sequence, and $x \in S(R^q)$ is an intermediate sequence. Then, the system $\Sigma_s : S(R^m) \to S(R^q)$ given by the recursion $x_{k+1} = f(x_k, u_k)$, $k = 0, 1, 2, \ldots$, is an input/state system, serving as the input/state part of Σ. Assume there is a reversible feedback function $\sigma : R^q \times R^m \to R^m$ for which the closed loop system $\Sigma_{s\sigma} : S(\theta^m) \to S(R^q)$ is stable. Using the function σ, we can construct a right coprime fraction representation for the input/state system Σ_s, as in (5.7). Let $\Sigma_s = P_s Q^{-1}$ be the resulting fraction representation, and note that its factorization space is given by $S(\theta^m)$, as before. But then, by the continuity of the function h, it follows that the system

(5.12) $\quad P := hP_s : S(\theta^m) \to S(R^p),$

given, for all $v \in S(\theta^m)$, by $Pv]_k = h(P_s v]_k)$, $k = 0, 1, 2, \ldots$, is a stable system, and

(5.13) $\quad \Sigma = PQ^{-1}.$

Thus, we have obtained a right fraction representation of the system Σ, which can be shown to be *right coprime* (HAMMER [1989c]). A continuous realization of the system P is given by

$$x_{k+1} = f(x_k, \sigma(x_k, v_k)), \qquad (5.14)$$
$$y_k = h(x_k),$$

where $v \in S(\theta^m)$ is the input of P, and $y = Pv$ is the output. A continuous realization of Q is described by (5.10). Thus, the numerator and denomintor of our right coprime fraction representation are both computable and implementable systems. We can summarize our discussion in the following

(5.15) THEOREM. Let $\Sigma : S(R^m) \to S(R^p)$ be a system having a continuous realization of the form (1.1), and let $\Sigma_s : S(R^m) \to S(R^q)$ be the input/state system induced by the recursion $x_{k+1} = f(x_k, u_k)$, where f is from (1.1). Let $\sigma : R^q \times R^m \to R^m$ be a reversible feedback function that stabilizes the system Σ_s. Then, the fraction representation $\Sigma = PQ^{-1} : S_u \to S(R^p)$ of (5.13) is right coprime, and has the factorization space $S(\theta^m)$. Furthermore, the systems $P : S(\theta^m) \to S(R^p)$ and $Q : S(\theta^m) \to S_u$ both possess continuous realizations, and Q is bicausal.

Notice that though the space S_u might be quite complicated, its computation is of no importance here. From the control theoretic point of view, only the factorization space of the coprime fraction representation is of importance, as we have discussed earlier. The space S_u, which forms the input space of the system Σ within the closed loop, is automatically generated by the closed loop system. To conclude, once a *reversible* stabilizing feedback function σ for the input/state part Σ_s of Σ is known, a right coprime fraction representation of the entire system Σ can be directly computed.

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