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Fraction representations of non-linear systems: a simplified approach

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A non-linear system Σ has a right fraction representation if it can be represented as $\Sigma = PQ^{-1}$, where P and Q are stable systems, and it has a left fraction representation if it can be represented as $\Sigma = G^{-1}T$, where G and T are stable systems. We develop here a theory of right and of left fraction representations for discrete-time non-linear systems with bounded input sequences. We indicate the connection between fraction representations and the stabilization problem for non-linear systems.

1. Introduction

In the present paper we provide a simplified version of the theory of right fraction representations and right coprimeness of non-linear systems developed in Hammer (1985 a), and we develop a theory of left fraction representations of non-linear systems. We restrict our attention to the case of discrete-time systems. The simplification in the theory of right fraction representations achieved in our present paper is the result of an assumption that we make concerning the conditions under which our systems operate. The assumption is that the systems are operated only by bounded input sequences, namely, that for each system Σ under consideration, there is a real number $\alpha > 0$ such that all input sequences to Σ are of amplitude α or less. This assumption, which is rather realistic from an engineering point of view and is satisfied in most practical applications, has a dramatic effect on the simplification of the proofs of some of the main results in the theory of fraction representations of non-linear systems. Under it, the proofs of our main results become relatively short and simple, and they utilize only standard rudimentary results regarding the topology of metric spaces, most of which can be found in the introductory text Kuratowski (1961). Indeed, it seems that the attempt in Hammer (1985 a) to attain uttermost generality and not to invoke our present assumption, carried us unnecessarily in several instances into major complications. Still, the basic concepts and techniques introduced in Hammer (1985 a) also form the basis of our present theory. We start with a discussion of our main motivation for studying fraction representations of non-linear systems.

Let Σ be a non-linear system. We say that Σ has a *right fraction representation* if there exists a pair of stable systems P and Q, where Q is invertible, such that $\Sigma = PQ^{-1}$. We say that Σ has a *left fraction representation* if there exists a pair of stable systems G and T where G is invertible, such that $\Sigma = G^{-1}T$. Right and left fraction representations play a fundamental role in the theory of stabilization of non-linear systems, as discussed in Hammer (1986). In order to describe the role of right and of left fraction representations in more detail, we briefly review some basic aspects of the stabilization theory developed in Hammer (1986).

Let Σ be a non-linear system. In order to stabilize the system Σ, we connect it in a closed-loop configuration of the classical form shown in Figure 1, where π is a causal

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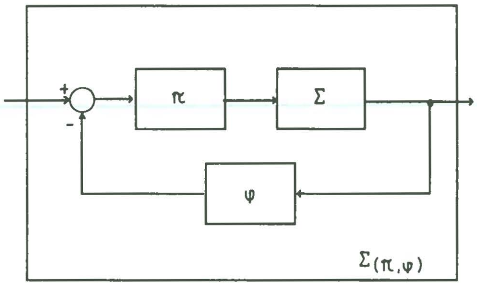

Figure 1.

precompensator and where φ is a causal feedback compensator. We denote by $\Sigma_{(\pi,\varphi)}$ the input/output relationship induced by the closed-loop system, and we assume that it is well defined. The closed-loop system is always well defined when Σ is a strictly causal system, so we assume for the moment that Σ is strictly causal (i.e. that Σ induces a delay of at least one step in the propagation of changes from its input to its output). As shown in Hammer (1986), it is particularly effective to choose the precompensator π and the feedback compensator φ in the form

$$\left.\begin{aligned}\pi &= B^{-1}\\ \varphi &= A\end{aligned}\right\} \tag{1.1}$$

where A and B are stable systems, where B is invertible, and where A and B^{-1} are causal. Assume further that Σ has a right fraction representation, namely, a fraction representation $\Sigma = PQ^{-1}$, where P and Q are stable systems and where Q is invertible. Then, using the compensators (1.1), and noting that, by the definition of a sum of systems, one has $I + APQ^{-1}B^{-1} = (BQ + AP)Q^{-1}B^{-1}$, the input/output relationship induced by the closed-loop system becomes (see, for example, Hammer 1984 b)

$$\begin{aligned}\Sigma_{(\pi,\varphi)} &= \Sigma\pi[1+\varphi\Sigma\pi]^{-1} = \Sigma B^{-1}[1 + A\Sigma B^{-1}]^{-1} = PQ^{-1}B^{-1}[1 + APQ^{-1}B^{-1}]^{-1}\\ &= PM^{-1}\end{aligned} \tag{1.2}$$

where

$$M := AP + BQ \tag{1.3}$$

is a stable and invertible system. Thus, if the stable systems A and B are chosen so that the inverse system M^{-1} is stable, then, by the stability of P, the composite system $\Sigma_{(\pi,\varphi)} = PM^{-1}$ becomes input/output stable. An invertible system M for which M and M^{-1} are both stable is called a *unimodular* system.

In view of the fact that $\Sigma_{(\pi,\varphi)} = PM^{-1}$, it follows that the system M influences the dynamical properties of the closed-loop system $\Sigma_{(\pi,\varphi)}$, and consequently, in practical situations, M is determined through the design objectives prescribed for the desired final system $\Sigma_{(\pi,\varphi)}$. Thus, M can be regarded as given, and the design procedure consists of the computation of an appropriate pair of stable systems A and B for which $AP + BQ = M$. This leads us to the following fundamental problem.

Coprimeness equation problem

Find all pairs of stable systems A, B satisfying the equation $AP + BQ = M$, where M is a specified unimodular system, and where P and Q originate from a right fraction representation $\Sigma = PQ^{-1}$ of the system Σ which needs to be stabilized.

Before continuing with our discussion, we remark that even though only input/output stability was mentioned in the previous paragraphs, a slight modification of the stability requirements imposed on the systems A and B of (1.1) will yield internal stability of the configuration shown in Figure 1. A detailed discussion of this point is provided in Hammer (1986), where the theory of internal stabilization is developed.

The resolution of the problem above involves the fundamental concept of right coprimeness introduced in Hammer (1985 a), and which we shall review in detail in § 3 of the present paper. Qualitatively speaking, a pair of stable systems P and Q (having in common their space of input sequences) is right coprime if, for every *un*bounded input sequence u, at least one of the output sequences Pu or Qu is unbounded (see § 3 for an accurate definition). For linear systems, this condition reduces to the requirement that P and Q have no unstable zeros in common. As one would expect from the analogy to the theory of linear systems, our problem is meaningful only when P and Q are right coprime, and, when P and Q are right coprime, one can find, for any unimodular system M (having the same input space as P and Q), a pair of stable systems A and B satisfying $AP + BQ = M$. This result, which is a cornerstone of our theory of non-linear systems, and which was discussed in Hammer (1985 a), will be discussed again in our present paper, and we shall provide here a new and simple proof of it. Considering that P and Q arise from the fraction representation $\Sigma = PQ^{-1}$ of the given system Σ, we conclude that Σ must possess a right coprime fraction representation, i.e., a representation of the form $\Sigma = PQ^{-1}$ where P and Q are stable and right coprime systems. Thus, our stabilization theory applies only to systems Σ possessing right coprime fraction representations.

In general, not every non-linear system Σ possesses a right coprime fraction representation. Nevertheless, as it turns out, most systems commonly encountered in applications do possess such fraction representations. Of basic significance to the theory of right coprime fraction representations is the concept of a homogeneous system. Qualitatively speaking, a system Σ is homogeneous if it behaves as a continuous map on sets of (bounded) input sequences which produce bounded output sequences. We show (§ 3 below and also Hammer 1985 a) that a system Σ possesses a right coprime fraction representation if and only if it is a homogeneous system. Fortunately, most systems appearing in nature are homogeneous. As an example of a rather large class of homogeneous systems, consider the following. A system Σ is called a *recursive system* if there is a pair of integers $\eta, \mu \geqslant 0$ and a function f such that, for every input sequence $u = (u_0, u_1, u_2, \ldots)$, the corresponding output sequence $y = (y_0, y_1, y_2, \ldots) = \Sigma u$ can be computed recursively in the form $y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu})$ for all integers $k = 0, 1, 2, \ldots$. The initial conditions $y_0, \ldots, y_\eta$ must, of course, be specified and fixed. The function f is called a *recursion function* of Σ. It can be shown (§ 3 below) that every recursive system having a *continuous* recursion function f is a homogeneous system. Recalling that homogeneous systems have right coprime fraction representations, this implies that the class of systems possessing right coprime fraction representations includes most systems encountered in engineering practice.

As we can infer from our brief discussion up to this point, the main objective of the theory of right coprimeness is to provide us with the means of constructing *one* pair of stable systems A, B satisfying $AP + BQ = M$, whenever P and Q are right coprime. The question of finding *all* pairs of stable systems A, B satisfying the equation $AP + BQ = M$ requires some further consideration. Crucial to the solution of this latter question is the theory of left fraction representations of non-linear systems developed in the present paper. It is rather easy to see how left fraction representations enter into the discussion, as follows.

Let Σ be a homogeneous system, let $\Sigma = PQ^{-1}$ be a right coprime fraction representation of Σ, and assume that Σ also has a left fraction representation $\Sigma = G^{-1}T$. Then, we clearly have $G^{-1}T = PQ^{-1}$, or

$$TQ = GP \tag{1.4}$$

Assume further that one pair of stable systems A, B satisfying $AP + BQ = M$ is known. To find additional pairs of stable systems A, B satisfying the same equation, we can proceed in a manner closely resembling linear methods. We choose an arbitrary stable system h, having appropriate input and output spaces, and we define the pair of stable systems

$$\left.\begin{aligned} A' &= A - hG \\ B' &= B + hT \end{aligned}\right\} \tag{1.5}$$

Then, by (1.4), we have $hTQ = hGP$, so that

$$A'P + B'Q = (A - hG)P + (B + hT)Q = AP + BQ + (hTQ - hGP) = AP + BQ = M$$

and we have obtained a new pair of stable systems A', B' satisfying $A'P + B'Q = M$. In fact, infinitely many pairs of stable systems A', B' satisfying $A'P + B'Q = M$ can be obtained in this way, one pair for each choice of h. Moreover, we shall see in § 4 that, using this simple method, one can actually obtain all solutions of the equation $A'P + B'Q = M$. Considering (1.2) and (1.3), we see from our discussion in the present paragraph that a theory of left fraction representations would be instrumental for the parametrization of the set of compensators stabilizing the system Σ through the configuration shown in Fig. 1. Thus, we may conclude that left fraction representations are of fundamental significance to the stabilization problem of non-linear systems just as much as they are of fundamental significance to the theory of stabilization of linear systems.

In § 4 we discuss the existence of left fraction representations, namely, of representations of the form $\Sigma = G^{-1}T$, where G and T are stable systems. We show there that a homogeneous system, in addition to having a right coprime fraction representation, as we have mentioned before, also has left fraction representations. In view of the previously mentioned fact that every recursive system with a continuous recursion function is a homogeneous system, we see that the theory developed in the present paper applies to most systems encountered in engineering applications.

We conclude this section with a few background remarks. The present paper is a continuation of the work reported in Hammer (1984 a, b, 1985 a, b, 1986). The theory of non-linear systems developed in these papers draws on basic ideas employed in the transfer matrix approach to linear system theory, as conceived in Rosenbrock (1970), in Desoer and Chan (1975), in Hammer (1983 a, b), in the references cited in these works, and in other related publications. Some recent alternative approaches to the study of non-linear systems can be found in Vidyasagar (1980), Sontag (1981), Desoer and Lin (1984), and the references cited in these papers.

2. Terminology and basics

Our discussion in this paper is stated within the mathematical framework developed in Hammer (1984 a, b, 1985 a, b, 1986). We devote the present section to a review and refinement of some basic aspects of this framework. As in the previous reports, our discussion is stated for the case of discrete-time systems. We remark that the theory can be extended to the case of continuous-time systems as well. We start by reviewing the spaces of input and output sequences of our systems.

Let R denote the set of real numbers. As usual, for an integer $m \geqslant 0$, we denote by R^m the set of all m-tuples of real numbers when $m > 0$, and we set $R^0 := 0$. We let $S_0(R^m)$ be the set of all infinite sequences of the form $u = \{u_0, u_1, \ldots, u_j, \ldots\}$, where $u_j \in R^m$ for all integers $j \geqslant 0$, and where the index j is interpreted as the time-marker. Given a sequence $u \in S_0(R^m)$ and an integer $i \geqslant 0$, we denote by u_i the ith element of the sequence. Given a pair of integers $j \geqslant i \geqslant 0$, we denote by u_i^j the elements $u_i, u_{i+1}, \ldots, u_j$ of the sequence. By a *system* Σ we simply mean a map $\Sigma : S_0(R^m) \to S_0(R^p)$, transforming input sequences of m-dimensional vectors into output sequences of p-dimensional vectors. We denote by Im Σ the set of all possible output sequences of the system Σ, and, for a subset $S \subset S_0(R^m)$, we denote by $\Sigma[S]$ the set of all possible output sequences of the restriction of Σ to S. In the space of sequences $S_0(R^m)$, we define the (standard) operation of addition elementwise so that, for every pair of sequences $u, v \in S_0(R^m)$, the *sum* sequence $w := u + v$ is given by $w_i = u_i + v_i$ for all integers $i \geqslant 0$. For a pair of systems $\Sigma_1, \Sigma_2 : S_0(R^m) \to S_0(R^p)$, the *sum* is the system $\Sigma := \Sigma_1 + \Sigma_2 : S_0(R^m) \to S_0(R^p)$ defined pointwise so that, for every input sequence $u \in S_0(R^m)$, the output sequence of Σ is $\Sigma u := \Sigma_1 u + \Sigma_2 u \in S_0(R^p)$. Given a system $\Sigma : S_0(R^m) \to S_0(R^p)$ and an input sequence $u \in S_0(R^m)$, we denote by $\Sigma u]_i =: y_i$ the ith element of the output sequence $y = \Sigma u$, and by $\Sigma u]_i^j$ the elements $y_i, y_{i+1}, \ldots, y_j$, where $j \geqslant i \geqslant 0$ are integers.

Of particular importance to our discussion are sets of bounded sequences, defined as follows. Let $\theta > 0$ be a real number. We denote by $[-\theta, \theta]^m$ the set of all m-dimensional real vectors $a = (a_1, \ldots, a_m)$ the components of which satisfy $|a_i| \leqslant \theta$, $i = 1, \ldots, m$. We denote by $S_0(\theta^m)$ the set of all sequences $u \in S_0(R^m)$ for which $u_i \in [-\theta, \theta]^m$ for all integers $i \geqslant 0$. Thus, $S_0(\theta^m)$ consists of all sequences bounded by θ. A sequence $u \in S_0(R^m)$ is said to be *bounded* if there is a real $\theta > 0$ such that $u \in S_0(\theta^m)$, and u is *unbounded* if no such θ exists. Adopting classical terminology, we say that a system $\Sigma : S_0(R^m) \to S_0(R^p)$ is *BIBO* (Bounded-Input Bounded-Output)-*stable* if, for every real $\theta > 0$, there is a real $M > 0$ such that $\Sigma[S_0(\theta^m)] \subset S_0(M^p)$. In other words, a BIBO-stable system is a system transforming bounded input sequences into bounded output sequences.

Next, we review our definition of stability. Following the spirit of the Liapunov definition of stability, we say, in qualitative terms, that a system is stable if it is BIBO-stable and if the map it induces is a continuous map. In order to study continuity of maps defined on our spaces of sequences, we induce a metric on these spaces as follows. Let $a = (a_1, \ldots, a_m)$ be an element of R^m. We denote $|a| := \max\{|a_1|, \ldots, |a_m|\}$. Given a sequence $u \in S_0(R^m)$, we denote $\rho(u) := \sup_{i \geqslant 0} \{2^{-i}|u_i|\}$. Using ρ, we define a metric on $S_0(R^m)$ given, for every pair of elements $u, v \in S_0(R^m)$, by $\rho(u, v) := \rho(u - v)$. Whenever discussing continuity, we shall always refer to continuity with respect to the topology induced by the metric ρ, unless explicitly stated otherwise. We can now define the notion of stability that we shall employ in our discussion. We say that a system $\Sigma : S_0(R^m) \to S_0(R^p)$ is *stable* if it is BIBO-stable and if, for every real $\theta > 0$, the restriction $\Sigma : S_0(\theta^m) \to S_0(R^p)$ is a continuous map.

As we have mentioned in § 1, we develop in the present paper a theory of fraction representations for systems Σ allowing only bounded input sequences. Thus, we shall assume throughout our discussion that the systems Σ whose fraction representations we derive, have the set $S_0(\alpha^m)$ as their domain, where $\alpha > 0$ is a fixed, but otherwise arbitrary, real number, so that $\Sigma : S_0(\alpha^m) \to S_0(R^p)$. In practical situations, the number α is determined, for instance, by the maximal input amplitude for which the mathematical model of Σ is valid, or by saturation effects of the system generating the inputs of Σ. Clearly, a system $\Sigma : S_0(\alpha^m) \to S_0(R^p)$ is stable if there is a real number $M > 0$ such that $\operatorname{Im} \Sigma \subset S_0(M^p)$ and if Σ is a continuous map. Of course, if Σ is not stable, its outputs may be unbounded.

Most of our examples as well as much of our motivation in developing the present theory are related to the study of recursive systems, so we provide now a formal definition of such systems. A system $\Sigma : S_0(R^m) \to S_0(R^p)$ is *recursive* if there exists a pair of integers $\eta, \mu \geqslant 0$ and a function $f : (R^p)^{\eta+1} \times (R^m)^{\mu+1} \to R^p$ such that, for every input sequence $u \in S_0(R^m)$, the output sequence $y := \Sigma u$ satisfies

$$y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu})$$

for all integers $k \geqslant 0$. Of course, the initial conditions $y_0, \ldots, y_\eta$ must be specified and fixed. The function f is called a *recursion function* of Σ. Our interest in recursive systems stems from the fact that they are among the systems most commonly encountered in engineering applications.

For the sake of completeness, we provide now a brief review of the standard notion of causality. A system $\Sigma : S_0(R^m) \to S_0(R^p)$ is *causal* (respectively, *strictly causal*) if it satisfies the following condition. For every integer $i \geqslant 0$, and for every pair of input sequences $u, v \in S_0(R^m)$ satisfying $u_0^i = v_0^i$, the output sequences satisfy $\Sigma u]_0^i = \Sigma v]_0^i$ (respectively, $\Sigma u]_0^{i+1} = \Sigma v]_0^{i+1}$).

We next provide a listing of a few standard results, taken from, e.g., Kuratowski (1961), and adapted to our present framework. We shall use these results repeatedly throughout our discussion without referencing.

Theorem 2.1

(i) For every real $\theta > 0$, the set $S_0(\theta^m)$, as well as any closed subset of it, is a compact set.

(ii) Let $\alpha, \beta > 0$ be a pair of real numbers, let $S \subset S_0(\alpha^m)$ be a closed subset, and let $F : S \to S_0(\beta^m)$ be a continuous function. Then, there is a continuous extension $F_x : S_0(\alpha^m) \to S_0(\beta^m)$ of F.

(iii) Let $\alpha, \beta > 0$ be a pair of real numbers, let $S \subset S_0(\alpha^m)$ be a closed subset, and let $F : S \to S_0(\beta^m)$ be a continuous function. If F is injective, then the restriction $F' : S \to \operatorname{Im} F$ is a homeomorphism, and $\operatorname{Im} F$ is a compact set.

Most of the results on fraction representations of a system Σ derived in our present paper are derived under the assumption that the system Σ is an injective system. We conclude this section with a discussion showing that the injectivity assumption is not really restrictive from the control theoretic point of view, in the sense that, through a minor modification of our basic control configuration (Fig. 1), the problem of stabilizing any strictly causal system can be transformed into a problem of stabilizing an injective system. This would imply that, from the stabilization point of view, it is enough to consider injective systems. Basically, we proceed as follows. We transform the given strictly causal system Σ into a new system Σ_e which is injective, and then we stabilize the injective system Σ_e using the configuration of Fig. 1. Due to the nature of the transformation, this will result in the stabilization of the original system Σ in a control configuration that is slightly different from Fig. 1. In qualitative terms, Σ_e is obtained by adding to Σ the identity system. We now construct Σ_e. Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be a strictly causal system. We denote $q := \max\{m, p\}$, and we define a pair of identity injections $\mathscr{I}_1$ and $\mathscr{I}_2$ as follows. If $m \leqslant p$, we identify $S_0(R^q) = S_0(R^p) = S_0(R^m) \times S_0(R^{q-m})$, we let $\mathscr{I}_1: S_0(R^p) \to S_0(R^q)$ be the identity map, and we let $\mathscr{I}_2: S_0(R^m) \to S_0(R^q)$ be the identity injection satisfying $\mathscr{I}_2[S_0(R^m)] = S_0(R^m) \times 0$, where the zero is the zero of the space $S_0(R^{q-m})$. If $m > p$, we identify $S_0(R^q) = S_0(R^m) = S_0(R^p) \times S_0(R^{q-p})$, we let $\mathscr{I}_1: S_0(R^p) \to S_0(R^q)$ be the identity injection satisfying $\mathscr{I}_1[S_0(R^p)] = S_0(R^p) \times 0$, where the zero is the zero of the space $S_0(R^{q-p})$, and we let $\mathscr{I}_2: S_0(R^m) \to S_0(R^q)$ be the identity map. We now define the system

$$\Sigma_e := \mathscr{I}_2 + \mathscr{I}_1 \Sigma: S_0(R^m) \to S_0(R^q) \tag{2.2}$$

and we note that this system is basically the sum of the original system Σ and the identity system, with appropriate formal adjustments related to the dimensions of the spaces.

The transformation which takes a strictly causal system Σ into the system Σ_e has a natural control theoretic interpretation. Indeed, when one uses the configuration of Fig. 1 to stabilize the system Σ_e, one also obtains stabilization of the system Σ in a slightly different control configuration, which qualitatively looks as the one in Fig. 2.

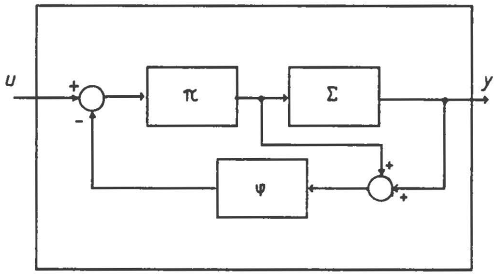

Figure 2.

Thus, instead of considering the stabilization of the system Σ, we can consider the stabilization of the system Σ_e. However, in view of the next lemma, the latter requires the stabilization of an injective system, and it follows that, from the stabilization theory point of view, we can restrict our attention to injective systems.

Lemma 2.3

Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be a strictly causal system, and let $\mathscr{I}_1$ and $\mathscr{I}_2$ be the two identity injections defined above. Then, the system $\Sigma_e := \mathscr{I}_2 + \mathscr{I}_1 \Sigma: S_0(R^m) \to S_0(R^q)$ (where $q = \max\{m, p\}$) is an injective system.

Proof

We slightly abuse the notation by denoting $\mathscr{I}_1 y = y$ and $\mathscr{I}_2 u = u$. Let $u, v \in S_0(R^m)$ be two sequences for which $\Sigma_e u = \Sigma_e v$. Now, by the strict causality of Σ, we have $\Sigma u]_0 = \Sigma v]_0$, so that, since $\Sigma_e u]_0 = u_0 + \Sigma u]_0 = \Sigma_e v]_0 = v_0 + \Sigma v]_0$, we obtain that $u_0 = v_0$. Preparing for induction, assume that $u]_0^k = v]_0^k$ for some integer $k \geqslant 0$. Then, by the strict causality of Σ, we have $\Sigma u]_{k+1} = \Sigma v]_{k+1}$, and, since $\Sigma_e u]_{k+1} = u_{k+1} + \Sigma u]_{k+1} = \Sigma_e v]_{k+1} = v_{k+1} + \Sigma v]_{k+1}$, we obtain that $u_{k+1} = v_{k+1}$. By induction this implies that $u_k = v_k$ for all integers $k \geqslant 0$, so that $u = v$, and Σ_e is injective. □

Finally, a notational remark. Given a map $P: S_1 \to S_2$, and a subset $S \subset S_2$, we denote by $P^*[S]$ the inverse image of the set S, namely, the set of all elements $u \in S_1$ satisfying $Pu \in S$.

3. Right fraction representations and coprimeness

In the present section we develop a theory of right fraction representations and coprimeness for non-linear systems $\Sigma: S_0(\alpha^m) \to S_0(R^p)$, where $\alpha > 0$ is a fixed, but otherwise arbitrary, real number. The results that we obtain here parallel the discussion of Hammer (1985 a), except that presently we restrict ourselves to systems allowing only inputs bounded by α, whereas in our previous report we considered systems with unbounded inputs. As mentioned in § 1, the restriction to systems with bounded inputs leads to a major simplification of our discussion, and, at the same time, it is a rather natural restriction from an engineering point of view. We say that a system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ has a *right fraction representation* if there is an integer $q > 0$, a subset $S \subset S_0(R^q)$, and a pair of stable maps $P: S \to \operatorname{Im} \Sigma$ and $Q: S \to S_0(\alpha^m)$, where Q is invertible, such that $\Sigma = PQ^{-1}$. We emphasize that we do not require that the systems P and Q of the fraction representation $\Sigma = PQ^{-1}$ have only bounded input sequences; the only assumption we make in this regard is that the original system Σ, the fraction representation of which we study, has $S_0(\alpha^m)$ as its domain of input sequences. Given a fraction representation $\Sigma = PQ^{-1}$, where $P: S \to \operatorname{Im} \Sigma$ and $Q: S \to S_0(\alpha^m)$, we call S the *factorization space* of the representation. (We remark that there is a slight abuse of notation in writing $\Sigma = PQ^{-1}$, since the codomain of Σ is $S_0(R^p)$ whereas the codomain of P is $\operatorname{Im} \Sigma$, but we adopt this notation for the sake of brevity.)

Now, let $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ be a system, and let $\Sigma = PQ^{-1}$ be a right fraction representation, where $P: S \to \operatorname{Im} \Sigma$ and $Q: S \to S_0(\alpha^m)$, and where the factorization space S is contained in $S_0(R^q)$. Also, let $M: S \to S$ be a stable map. The main question that we consider in the present section is under what conditions can one find a pair of stable maps $A: \operatorname{Im} \Sigma \to S_0(R^q)$ and $B: S_0(\alpha^m) \to S_0(R^q)$ satisfying the equation

$$AP + BQ = M \tag{3.1}$$

Using the insight that we gained from the linear theory, we expect that the possibility of solving (3.1) for an arbitrary stable map $M: S \to S$ would involve a certain condition of 'right coprimeness' of the maps P and Q. In Hammer (1985 a) we saw that a natural definition of the notion of right coprimeness in our present situation is as follows.

Definition 3.2

Let $S \subset S_0(R^q)$ be a subset. Two stable maps $P: S \to S_0(R^p)$ and $Q: S \to S_0(R^m)$ are *right coprime* if the following conditions hold:

(i) For every real $\tau > 0$ there exists a real $\theta > 0$ such that

$$P^*[S_0(\tau^p)] \cap Q^*[S_0(\tau^m)] \subset S_0(\theta^q)$$

(ii) For every real $\tau > 0$, the set $S \cap S_0(\tau^q)$ is a closed subset of $S_0(\tau^q)$.

In intuitive terms, condition (i) of Definition 3.2 means that, for every unbounded input sequence $u \in S$, at least one of the output sequences Pu or Qu is unbounded. Condition (ii) of the definition is a natural topological requirement, essential for the technical viability of the concept of right coprimeness. In the linear case, condition (i) reduces to the requirement that P and Q have no unstable zeros in common, whereas condition (ii) holds automatically by linearity. We say that the system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ has a *right coprime fraction representation* if it has a right fraction representation $\Sigma = PQ^{-1}$ in which the stable systems P and Q are right coprime. In Theorem 3.4 below we show that, when $\Sigma = PQ^{-1}$ is a right coprime fraction representation, then the coprimeness equation problem in § 1 has at least one solution. Before that, we discuss some properties of right coprime fraction representations. First, some terminology. Let $M: S_1 \to S_2$ be a map, where $S_1 \subset S_0(R^m)$ and $S_2 \subset S_0(R^p)$. We say that M is a *unimodular map* if M is invertible and if M and M^{-1} are both stable maps. If there is a unimodular map $M: S_1 \to S_2$, then we say that the spaces S_1 and S_2 are *S-morphic* (stability-morphic). Now, let $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ be an injective system, and assume it has a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P: S \to \operatorname{Im} \Sigma$ and $Q: S \to S_0(\alpha^m)$, and where $S \subset S_0(R^q)$. In view of the injectivity of Σ, the map P is injective and, since it is evidently also surjective, it follows that P is invertible. It is easy to see intuitively that P^{-1} must be BIBO-stable. Indeed, assume there is a bounded sequence $u \in \operatorname{Im} \Sigma$ for which $P^{-1}u$ is not bounded. Denoting $v := P^{-1}u$ and $w := Qv$, we evidently have that $w \in S_0(\alpha^m)$. But then, we have an unbounded sequence v for which both of the output sequences $Pv = u$ and $Qv = w$ are bounded, contradicting the fact that P and Q are right coprime. Thus, for every bounded sequence $u \in \operatorname{Im} \Sigma$, the sequence $P^{-1}u$ must also be bounded. In fact, we show in the next proposition that the map P^{-1} must actually be stable, so that, in a right coprime fraction representation of a system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$, the numerator map P is always unimodular. This fact is a clear departure from the analogy to the theory of fraction representations of linear systems, and it is a consequence of the assumption that the input space of the system Σ is bounded. In the linear case, it is not possible to assume that the input space of the system is bounded, since this would violate the linearity of the space. Thus we see that, in a sense, the non-linear theory is simpler than the linear theory, and it allows us to take advantage of the actual conditions under which the system Σ operates.

Proposition 3.3

Let $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ be an injective system, and assume it has a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P: S \to \operatorname{Im} \Sigma$ and $Q: S \to S_0(\alpha^m)$, and where $S \subset S_0(R^q)$. Then, the map $P^{-1}: \operatorname{Im} \Sigma \to S$ is a stable map.

Proof

Taking $\tau \geqslant \alpha$ in condition (i) of Definition 3.2 of the concept of right coprimeness, we obtain that there is a real $\theta_\tau > 0$ such that $P^*[S_0(\tau^p)] \cap Q^*[S_0(\tau^m)] \subset S_0((\theta_\tau)^q)$. But then, since actually $Q^*[S_0(\tau^m)] = S$ for $\tau \geqslant \alpha$, this implies that $P^*[S_0(\tau^p)] \subset S_0((\theta_\tau)^q)$. By the stability of P, the restriction of P to $S \cap S_0((\theta_\tau)^q)$ is a continuous function, and whence, by the closure of $S_0(\tau^p)$, the set $S_\tau := P^*[S_0(\tau^p)]$ is a closed subset of $S \cap S_0((\theta_\tau)^q)$. Moreover, since by condition (ii) of Definition 3.2 the set $S \cap S_0((\theta_\tau)^q)$ is a closed subset of $S_0((\theta_\tau)^q)$, it follows that S_τ also is a closed subset of $S_0((\theta_\tau)^q)$, and whence S_τ is a compact set. Further, by the injectivity of Σ, the map P is injective as well, and consequently, by the continuity of P and the compactness of S_τ, the restriction $P:S_\tau \to (\mathrm{Im}\,\Sigma)\cap S_0(\tau^p)$ is a homeomorphism, and whence the map $P^{-1}:(\mathrm{Im}\,\Sigma)\cap S_0(\tau^p)\to S_\tau \subset S_0((\theta_\tau)^q)$ is continuous for every real $\tau \geqslant \alpha$. But, since evidently $S_0(\tau^p)\subset S_0(\alpha^p)$ whenever $\tau<\alpha$, this implies that, for every real $\tau>0$, there is a real $\theta>0$ such that $P^{-1}[(\mathrm{Im}\,\Sigma)\cap S_0(\tau^p)]\subset S_0((\theta)^q)$ and that the restriction $P^{-1}:(\mathrm{Im}\,\Sigma)\cap S_0(\tau^p)\to S_0(\theta^q)$ is a continuous map. Thus, P^{-1} is stable, and our proof is concluded. □

Using proposition 3.3, we can easily construct a solution of the equation $AP+BQ=M$. Indeed, let $\Sigma:S_0(\alpha^m)\to S_0(R^p)$ be an injective system, and assume it has a right coprime fraction representation $\Sigma=PQ^{-1}$, where $P:S\to\mathrm{Im}\,\Sigma$ and $Q:S\to S_0(\alpha^m)$ and where $S\subset S_0(R^q)$, and let $M:S\to S$ be any stable map. Then, in view of Proposition 3.3, the map $A:=MP^{-1}:\mathrm{Im}\,\Sigma\to S_0(R^q)$ is stable. Letting $B:=0:S_0(\alpha^m)\to S_0(R^q)$ be the constant zero map, which is evidently stable, we have $AP+BQ=MP^{-1}P+0=M$, and we constructed a solution of the equation $AP+BQ=M$. We note that this particular solution is not applicable to the stabilization procedure outlined in (1.1)–(1.3) since the map B here is not invertible and A here may not be causal. Nevertheless, this solution is important since, as we show in § 4, once one solution of the equation $AP+BQ=M$ is known, all other solutions can be constructed through the straightforward procedure outlined in (1.5). We summarize our discussion in this paragraph in the following result.

Theorem 3.4

Let $\Sigma:S_0(\alpha^m)\to S_0(R^p)$ be an injective system, and assume it has a right fraction representation $\Sigma=PQ^{-1}$, where $P:S\to\mathrm{Im}\,\Sigma$ and $Q:S\to S_0(\alpha^m)$, and where $S\subset S_0(R^q)$. If P and Q are right coprime, then for every stable map $M:S\to S$, there exists a pair of stable maps $A:\mathrm{Im}\,\Sigma\to S_0(R^q)$ and $B:S_0(\alpha^m)\to S_0(R^q)$ satisfying $AP+BQ=M$.

Of course, in our discussion in the previous paragraph, we assumed that a right coprime fraction representation $\Sigma=PQ^{-1}$ of the given system Σ is known. As we shall see shortly, the construction of right coprime fraction representations for a system $\Sigma:S_0(\alpha^m)\to S_0(R^p)$ is fairly simple, when they exist. We next discuss the basic problem of the existence of right coprime fraction representations.

Of fundamental significance to the theory of right coprime fraction representations of non-linear systems is the concept of a homogeneous system, which, qualitatively speaking, is a system that behaves as a continuous map whenever its outputs are bounded. The exact definition is as follows.

Definition 3.5

A system $\Sigma:S_0(\alpha^m)\to S_0(R^p)$ is a homogeneous system if, for every subset $S\subset S_0(\alpha^m)$ for which there exists a real $\theta>0$ such that $\Sigma[S]\subset S_0(\theta^p)$, the restriction of Σ to the closure $\bar{S}$ of S is a continuous map $\Sigma:\bar{S}\to S_0(\theta^p)$.

The significance of the class of homogeneous systems comes from the fact that it is identical to the class of systems possessing right coprime fraction representations, as stated in the next result.

Theorem 3.6

An injective system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ has a right coprime fraction representation if and only if it is a homogeneous system.

Proof

Assume first that the injective system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ has a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P: S \to \text{Im}\,\Sigma$ and $Q: S \to S_0(\alpha^m)$, and where $S \subset S_0(R^q)$. Now, let $S_1 \subset S_0(\alpha^m)$ be any subset for which there exists a real $\theta > 0$ such that $\Sigma[S_1] \subset S_0(\theta^p)$. Denote $S' := \Sigma[S_1]$, and let $\bar{S}'$ be the closure of S' in $S_0(\theta^p)$. By the stability of P^{-1} derived in Proposition 3.3, it follows that there is a real $\tau > 0$ such that $P^*[\bar{S}'] \subset S_0(\tau^q)$. By the stability of the map P, the restriction $P: S \cap S_0(\tau^q) \to S_0(R^p)$ is continuous (and bounded), so the set $S'' := P^*[\bar{S}']$ is a closed subset of $S \cap S_0(\tau^q)$. But then, by condition (ii) of Definition 3.2, the set S'' is a closed subset of $S_0(\tau^q)$, and whence is compact. Consequently, by the continuity and injectivity of the map Q, the restriction $Q: S'' \to Q[S'']$ is a homeomorphism, so $Q^{-1}: Q[S''] \to S''$ is continuous and the set $Q[S'']$ is compact. Moreover, since clearly $S_1 \subset Q[S'']$, the latter implies that also $\bar{S}_1 \subset Q[S'']$, where $\bar{S}_1$ is the closure of S_1 in $S_0(\alpha^m)$, and it follows that the restriction $Q^{-1}: \bar{S}_1 \to Q^{-1}[\bar{S}_1]$ is a continuous map. Thus, since $\Sigma = PQ^{-1}$ and P is stable, we obtain that the restriction $\Sigma: \bar{S}_1 \to \bar{S}' \subset S_0(\theta^p)$ is a continuous map, and Σ is a homogeneous system.

Conversely, assume that Σ is a homogeneous system. Then, a right coprime fraction representation of Σ can be simply constructed as follows. Define $P: \text{Im}\,\Sigma \to \text{Im}\,\Sigma$ as $P := I$, the identity map, and $Q: \text{Im}\,\Sigma \to S_0(\alpha^m)$ as $Q := \Sigma^{-1}$, the inverse system (recall that Σ is injective, so $\Sigma: S_0(\alpha^m) \to \text{Im}\,\Sigma$ is a set isomorphism). The map P is clearly stable; to show that Q is stable, we proceed as follows. First, Q is evidently BIBO-stable since $\text{Im}\,Q \subset S_0(\alpha^m)$. To show that Q is also continuous, let $\theta > 0$ be an arbitrary real number, and denote $S^* := Q[(\text{Im}\,\Sigma) \cap S_0(\theta^p)] \subset S_0(\alpha^m)$. Then, $\Sigma[S^*] = \Sigma Q[(\text{Im}\,\Sigma) \cap S_0(\theta^p)] = \Sigma\Sigma^{-1}[(\text{Im}\,\Sigma) \cap S_0(\theta^p)] = (\text{Im}\,\Sigma) \cap S_0(\theta^p) \subset S_0(\theta^p)$, so that, by homogeneity, the restriction $\Sigma: \bar{S}^* \to S_0(\theta^p)$ is a continuous map. But then, by the compactness of $\bar{S}^* \subset S_0(\alpha^m)$ and the injectivity of Σ, it follows that the restriction $\Sigma: \bar{S}^* \to \Sigma[\bar{S}^*]$ actually is a homeomorphism, so that $\Sigma^{-1}: \Sigma[\bar{S}^*] \to \bar{S}^*$ is continuous, and $\Sigma[\bar{S}^*]$ is a compact set. Further, since $\Sigma[\bar{S}^*]$ is contained in $S_0(\theta^p)$, we have $\Sigma[\bar{S}^*] \subset (\text{Im}\,\Sigma) \cap S_0(\theta^p)$. Considering that $\Sigma[\bar{S}^*] \supset \Sigma[S^*] = (\text{Im}\,\Sigma) \cap S_0(\theta^p)$, it follows that $\Sigma[\bar{S}^*] = (\text{Im}\,\Sigma) \cap S_0(\theta^p)$. Finally, since $\Sigma^{-1}: \Sigma[\bar{S}^*] \to \bar{S}^*$ is continuous, the restriction $Q\,(=\Sigma^{-1}): (\text{Im}\,\Sigma) \cap S_0(\theta^p) \to S_0(\alpha^m)$ is continuous, and Q is a stable map. Thus, since clearly $PQ^{-1} = I(\Sigma^{-1})^{-1} = \Sigma$ and P, Q are both stable, we constructed a right fraction representation $\Sigma = PQ^{-1}$. We next show that this fraction representation is coprime.

The factorization space of the above representation $\Sigma = PQ^{-1}$ is $S = \text{Im}\,\Sigma$; to show that condition (ii) of Definition 3.2 holds, we note that since $(\text{Im}\,\Sigma) \cap S_0(\theta^p) = \Sigma[\bar{S}^*]$ is a compact set, the set $S \cap S_0(\theta^p) = (\text{Im}\,\Sigma) \cap S_0(\theta^p)$ is evidently a closed subset of $S_0(\theta^p)$. To show that condition (i) of Definition 3.2 holds as well, we note that, for every real $\tau > 0$, $P^*[S_0(\tau^p)] \cap Q^*[S_0(\tau^m)] = \{(\text{Im}\,\Sigma) \cap S_0(\tau^p)\} \cap \Sigma[S_0(\tau^m) \cap S_0(\alpha^m)] \subset (\text{Im}\,\Sigma) \cap S_0(\tau^p) \subset S_0(\tau^p)$, and condition (i) of Definition 3.2 holds with $\theta = \tau$. Thus, P and Q are right coprime, and our proof is concluded. □

The previous proof provides us with a very simple method of constructing a right coprime fraction representation of an injective homogeneous system, and we summarize this fact in the following.

Corollary 3.7

Let $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ be an injective homogeneous system. Then, the maps $P := I: \mathrm{Im}\,\Sigma \to \mathrm{Im}\,\Sigma$, the identity map, and $Q := \Sigma^{-1}: \mathrm{Im}\,\Sigma \to S_0(\alpha^m)$ induce a right coprime fraction representation $\Sigma = PQ^{-1}$. The factorization space of this representation is $\mathrm{Im}\,\Sigma$.

As we can see, the construction of a right coprime fraction representation for an injective homogeneous system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ is a fairly simple matter. From Theorem 3.6, we know that homogeneous systems are the only injective systems possessing right coprime fraction representations. Finally, we recall from our discussion at the end of § 2 that, from the control-theoretic point of view, the restriction to injective systems does not significantly impair the applicability of our theory to the study of stabilization of non-injective systems. Our next objective is to provide some examples of homogeneous systems. In fact, in the next statement we show that most systems encountered in common engineering practice are homogeneous systems.

Proposition 3.8

Let $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ be a recursive system. If Σ has a continuous recursion function, then it is a homogeneous system.

Proof

We use in this proof uniform continuity, since all our domains are compact. Let $y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu})$ be a recursive representation of Σ, where $f: (R^p)^{\eta+1} \times ([-\alpha, \alpha]^m)^{\mu+1} \to R^p$ is a continuous function. We define recursively the following functions F_j, $j = 0, 1, 2, \ldots$. For $j = 0, \ldots, \eta$, we let $F_j := y_j$, where $y_0, \ldots, y_\eta$ are the fixed initial conditions of the system; for $j > \eta$, we set $F_{\eta+k+1}: ([-\alpha, \alpha]^m)^{k+\mu+1} \to R^p$ to be

$$F_{\eta+k+1}(u_0, \ldots, u_{\mu+k}) := f(F_k, \ldots, F_{k+\eta}, u_k, \ldots, u_{k+\mu})$$

$k = 0, 1, 2, \ldots$, where, in the last formula, we have omitted the variables of the functions $F_k, \ldots, F_{k+\eta}$. Clearly, for every input sequence $u \in S_0(\alpha^m)$, we have that $F_{\eta+k+1}(u_0, \ldots, u_{\mu+k}) = y_{k+\eta+1}$, where $y = \Sigma u$ is the output sequence. By the continuity of the function f, it follows that each one of the functions F_j, $j = 0, 1, 2, \ldots$, is a continuous function over its entire space of definition. Now, let $S \subset S_0(\alpha^m)$ be a subset for which there is a real $\theta > 0$ such that $\Sigma[S] \subset S_0(\theta^p)$. In order to show that Σ is a homogeneous system, we show that (i) $\Sigma[\bar{S}] \subset S_0(\theta^p)$, where $\bar{S}$ is the closure of S in $S_0(\alpha^m)$, and that (ii) the restriction $\Sigma: \bar{S} \to S_0(\theta^p)$ is a continuous map. To prove (i), let $u^1, u^2, \ldots$, be a sequence of elements of the set S converging to a point $u^* \in S_0(\alpha^m)$. We have to show that the output sequence $y^* := \Sigma u^*$ also belongs to $S_0(\theta^p)$. In view of the fact that the sequence $\{u^i\}$ converges to u^*, we have that, for any integer $k \geqslant 0$, the sequence $(u_0^i, \ldots, u_{\mu+k}^i)$, $i = 1, 2, \ldots$, converges to $(u_0^*, \ldots, u_{\mu+k}^*)$ so that, by the continuity of the function $F_{\eta+k+1}$, we obtain $y_{k+\eta+1}^* = F_{\eta+k+1}(u_0^*, \ldots, u_{\mu+k}^*) = \lim_{i\to\infty} F_{\eta+k+1}(u_0^i, \ldots, u_{\mu+k}^i)$. But then, since $\Sigma u^i \in S_0(\theta^p)$ for all integers $i \geqslant 1$, we have that $F_{\eta+k+1}(u_0^i, \ldots, u_{\mu+k}^i) \in [-\theta, \theta]^p$ for all integers $i \geqslant 1$, and it follows that $y_{k+\eta+1}^* \in [-\theta, \theta]^p$ for all integers $k \geqslant 0$. Since the output values $y_0, \ldots, y_\eta$ are the same for all input sequences, this implies that $\Sigma u^* \in S_0(\theta^p)$, and so $\Sigma[\bar{S}] \subset S_0(\theta^p)$, and (i) holds.

To prove (ii), let $\varepsilon > 0$ be a real number, and let $n > \eta$ be an integer such that

$(2^{-n})\theta < \varepsilon$. By the continuity of the functions $F_{\eta+k+1}$, there is, for each integer $k \geqslant 0$, a real number $\delta_k > 0$ such that $|F_{\eta+k+1}(u_0, \ldots, u_{\mu+k}) - F_{\eta+k+1}(v_0, \ldots, v_{\mu+k})| < \varepsilon$ whenever $|(u_0, \ldots, u_{\mu+k}) - (v_0, \ldots, v_{\mu+k})| < \delta_k$. Denoting $a := n - \eta - 1$, $b := a + \mu$, $\gamma_k := (2^{-(\eta+k+1)})|F_{\eta+k+1}(u_0, \ldots, u_{\mu+k}) - F_{\eta+k+1}(v_0, \ldots, v_{\mu+k})|$, and $\delta := (2^{-b}) \min \{\delta_0, \ldots, \delta_a\} > 0$, it follows by the definition of our metric ρ that, for any pair of sequences $u, v \in \bar{S}$ satisfying $\rho(u - v) < \delta$, one has $\rho(\Sigma u - \Sigma v) \leqslant \max \{\gamma_0, \gamma_1, \ldots, \gamma_a, 2(2^{-(n+1)})\theta\} < \varepsilon$ by our choice of δ and n. Thus, for every real $\varepsilon > 0$, there is a real $\delta > 0$ such that $\rho(\Sigma u - \Sigma v) < \varepsilon$ for all pairs of sequences $u, v \in \bar{S}$ satisfying $\rho(u - v) < \delta$, and the restriction of Σ to $\bar{S}$ is continuous. □

Considering the fact that most systems encountered in engineering practice are recursive systems having continuous recursion functions, it follows from Proposition 3.8 that the theory of fraction representations and coprimeness that we develop in the present paper is of fairly wide applicability. Moreover, the class of homogeneous systems also includes systems which are not necessarily recursive, like the class of all systems possessing a continuous realization, as described in the following remark.

Remark 3.9

A system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ has a *continuous realization* if there is a pair of continuous functions $f: (R^n)^{\eta+1} \times ([-\alpha, \alpha]^m)^{\mu+1} \to R^n$ and $h: (R^n) \times ([-\alpha, \alpha]^m) \to R^p$ such that, for every input sequence $u \in S_0(\alpha^m)$, the output sequence $y := \Sigma u$ can be computed from the relations $v_{k+\eta+1} = f(v_k, \ldots, v_{k+\eta}, u_k, \ldots, u_{k+\mu})$, $y_k = h(v_k, u_k)$, $k = 0, 1, 2, \ldots$. Here, v is an intermediate sequence, and the initial conditions $v_0, \ldots, v_\eta$ are fixed and given. Using an argument similar to the one proving Proposition 3.8, it is easy to see that every system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ having a continuous realization is a homogeneous system.

Returning for a moment to our discussion of the implications of the injectivity assumption in § 2, we saw in Lemma 2.3 that the system Σ_e of (2.2) is an injective system whenever the original system Σ is a strictly causal system. We next show that when Σ is a homogeneous system, so also is Σ_e. Thus, when Σ is a strictly causal homogeneous system, the system Σ_e is injective and homogeneous. This shows that the theory of fraction representations developed in our present paper is applicable to Σ_e, and thus, modulo the simple transformation leading from Σ to Σ_e, it is applicable to most practical systems.

Proposition 3.10

Let $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ be a strictly causal homogeneous system, and let $\Sigma_e := \mathscr{I}_2 + \mathscr{I}_1 \Sigma: S_0(\alpha^m) \to S_0(R^q)$ (where $q = \max \{m, p\}$) be the system constructed in (2.2) from Σ. Then, Σ_e is an injective and homogeneous system.

Proof

Let $S \subset S_0(\alpha^m)$ be a subset for which there is a real number $\theta > 0$ such that $\Sigma_e[S] \subset S_0(\theta^q)$, and let $\bar{S}$ be the closure of S in $S_0(\alpha^m)$. In order to show that Σ_e is homogeneous, we have to show that the restriction $\Sigma_e: \bar{S} \to S_0(\theta^q)$ is a continuous function. Now, since $\mathscr{I}_1 \Sigma = \Sigma_e - \mathscr{I}_2$, and since $S \subset S_0(\alpha^m)$ and $\Sigma_e[S] \subset S_0(\theta^q)$, it follows that $\Sigma[S] \subset S_0((\theta + \alpha)^p)$, and whence, by the homogeneity of Σ, the restriction

$\Sigma:\bar{S}\to S_0((\theta+\alpha)^p)$ is a continuous map. But then, by the evident continuity of the identity injections $\mathscr{I}_1$ and $\mathscr{I}_2$, we obtain that the restriction of Σ_e to $\bar{S}$ is continuous as well, and our proof is concluded. □

We conclude the present section with a discussion of the uniqueness of right coprime fraction representations, showing that right coprime fraction representations are uniquely determined, up to a unimodular transformation. This is, of course, a consequence of Proposition 3.3.

Theorem 3.11

Let $\Sigma:S_0(\alpha^m)\to S_0(R^p)$ be an injective homogeneous system, and let $\Sigma=PQ^{-1}$ and $\Sigma=P'Q'^{-1}$ be two right coprime fraction representations of Σ, with factorization spaces $S, S'\subset S_0(R^q)$, respectively. Then, there is a unimodular map $M:S'\to S$ such that $P'=PM$ and $Q'=QM$.

Proof

By the injectivity of Σ, the maps $P:S\to\operatorname{Im}\Sigma$ and $P':S'\to\operatorname{Im}\Sigma$ are both set isomorphisms, and whence P^{-1} and P'^{-1} exist. We define $M:=P^{-1}P':S'\to S$, and we show that M is unimodular. Since M is evidently invertible, we only have to show that $M=P^{-1}P'$ and $M^{-1}=P'^{-1}P$ are both stable. But, this is a direct consequence of the fact that, by Proposition 3.3, the maps P^{-1} and P'^{-1} are both stable maps. Also, clearly $P'=PM$, and, since $Q'=(Q'P'^{-1})P'=(QP^{-1})P'$ by the equality $PQ^{-1}=P'Q'^{-1}$, we have $Q'=Q(P^{-1}P')=QM$, and our proof is concluded. □

Finally, combining Proposition 3.7 and Theorem 3.11, we obtain the following characterization of the factorization space of a right coprime fraction representation.

Theorem 3.12

Let $\Sigma:S_0(\alpha^m)\to S_0(R^p)$ be an injective homogeneous system. Then, the factorization space of any right coprime fraction representation of Σ is S-morphic to $\operatorname{Im}\Sigma$.

In Summary, we have seen in this section that it is rather simple to develop a theory of right coprime fraction representations for systems $\Sigma:S_0(\alpha^m)\to S_0(R^p)$ which are homogeneous and injective. We have seen that the class of homogeneous systems includes most systems of engineering interest, and at the end of § 2 and in Proposition 3.10, we showed that the injectivity assumption does not significantly impair the applicability of our results to the study of stabilization of non-injective systems.

4. Left fraction representations

In the present section we develop a theory of left fraction representations for non-linear systems $\Sigma:S_0(\alpha^m)\to S_0(R^p)$. Our main objective in this discussion is to provide a means of parametrizing the set of all solutions A, B of the equation $AP+BQ=M$, where M is a fixed stable map, and where P and Q arise from a right coprime fraction representation $\Sigma=PQ^{-1}$ of the given system Σ. We have discussed this problem in § 1, and we have provided an ourline of the basic idea of our solution in (1.5).

Let $\Sigma:S_0(\alpha^m)\to S_0(R^p)$ be a non-linear system. We say that Σ has a *left fraction representation* if there is an integer $q>0$, a subspace $S\subset S_0(R^q)$, and a pair of stable maps $G:\mathrm{Im}\,\Sigma \to S$ and $T: S_0(\alpha^m) \to S$, where G is invertible, such that $\Sigma = G^{-1}T$. The space S is called the *factorization space* of the fraction representation $\Sigma = G^{-1}T$. We adopt the convention of *always* taking S to be Im T, so that T is always a surjective map. Then, when Σ is an injective system, T is both injective and surjective, and thus possesses an inverse T^{-1}. (Again, as in § 3, we slightly abuse the notation by writing $\Sigma = G^{-1}T$, since the codomain of Σ is $S_0(R^p)$ whereas the codomain of G^{-1} is Im Σ. We adopt this notation for the sake of brevity.)

As it turns out, the theory of left fraction representations for a system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ is extremely simple, due to the compactness of the domain $S_0(\alpha^m)$ of the system. In fact, if Σ has a left fraction representation, then all left fraction representations of Σ are actually left 'coprime' fraction representations in an intuitive sense. The origin of this fact is the following result.

Proposition 4.1

Let $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ be an injective system, and assume it has a left fraction representation $\Sigma = G^{-1}T$, where $G:\mathrm{Im}\,\Sigma \to S$ and $T: S_0(\alpha^m) \to S$, and where $S \subset S_0(R^q)$. Then, the map $T^{-1}: S \to S_0(\alpha^m)$ is a stable map.

Proof

The map T^{-1} is BIBO-stable, since evidently $T^{-1}[S_0(\theta^q) \cap S] \subset T^{-1}[S] \subset S_0(\alpha^m)$ for every real $\theta > 0$. Thus, it only remains to show that the restriction $T^{-1}: S_0(\theta^q) \cap S \to S_0(\alpha^m)$ is a continuous map. In view of the fact that Im $T = S$, it follows by the continuity of T that the set $S' := T^{-1}[S_0(\theta^q) \cap S]$ is a closed subset of $S_0(\alpha^m)$, the domain of T, and whence S' is compact. Considering the injectivity of T this implies, again by continuity, that the restriction $T: S' \to S_0(\theta^q) \cap S$ is a homeomorphism, and, consequently, the restriction $T^{-1}: S_0(\theta^q) \cap S \to S' \subset S_0(\alpha^m)$ is a continuous map. □

Proposition 4.1 actually means that, in any left fraction representation $\Sigma = G^{-1}T$ of an injective system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$, the numerator T is a unimodular map, since T and T^{-1} are both stable. Thus, we have a complete description of the structure of left fraction representations of injective systems. The fact that T is always unimodular forms a major departure from the analogy to the theory of fraction representations of linear systems. It is a consequence of the compactness of the domain $S_0(\alpha^m)$ of the system Σ. Proposition 4.1 also provides us with a complete characterization of the class of injective systems $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ possessing left coprime fraction representations. Indeed, when Σ is injective, then the restriction $\Sigma: S_0(\alpha^m) \to \mathrm{Im}\,\Sigma$ is a set isomorphism, so $\Sigma^{-1}: \mathrm{Im}\,\Sigma \to S_0(\alpha^m)$ exists. Now, if there is a left fraction representation $\Sigma = G^{-1}T$, then $\Sigma^{-1} = T^{-1}G$, and it follows by Proposition 4.1 and the stability of G that Σ^{-1} is a stable system. Conversely, if Σ^{-1} is a stable system, we have that $G' := \Sigma^{-1}: \mathrm{Im}\,\Sigma \to S_0(\alpha^m)$ is a stable map, and, setting $T' := I: S_0(\alpha^m) \to S_0(\alpha^m)$, the identity system, we obtain the left fraction representation $\Sigma = G'^{-1}T'$, having the factorization space Im Σ. This proves the following result.

Theorem 4.2

An injective system $\Sigma: S_0(\alpha^m) \to S_0(R^p)$ has a left fraction representation if and only if $\Sigma^{-1}: \mathrm{Im}\,\Sigma \to S_0(\alpha^m)$ is a stable system.

From our experience with the theory of linear systems, it would seem that the condition of Theorem 4.2, namely, the requirement that Σ^{-1} be a stable system, is rather restrictive. However, this is not true in the non-linear case, and the class of injective non-linear systems $\Sigma: S_0(\alpha^m) \rightarrow S_0(R^p)$ for which Σ^{-1} is a stable system actually is a large class of systems. For instance, it is a direct consequence of Corollary 3.7 that every injective homogeneous system $\Sigma: S_0(\alpha^m) \rightarrow S_0(R^p)$ satisfies the requirement that $\Sigma^{-1}: \operatorname{Im} \Sigma \rightarrow S_0(\alpha^m)$ be a stable system. Thus, the class of systems possessing left fraction representations includes all systems having right coprime fraction representations. We state this fact in the following result.

Theorem 4.3

An injective homogeneous system $\Sigma: S_0(\alpha^m) \rightarrow S_0(R^p)$ has a left fraction representation.

Considering our discussion in the paragraph preceding Theorem 4.2, we see that a left fraction representation $\Sigma = G^{-1}T$ for an injective homogeneous system $\Sigma: S_0(\alpha^m) \rightarrow S_0(R^p)$ can be obtained simply by setting $G := \Sigma^{-1}: \operatorname{Im} \Sigma \rightarrow S_0(\alpha^m)$ and $T := I: S_0(\alpha^m) \rightarrow S_0(\alpha^m)$, the identity map.

Another direct consequence of Theorem 4.3 is that, for every strictly causal homogeneous system $\Sigma: S_0(\alpha^m) \rightarrow S_0(R^p)$, the system Σ_e of (2.2) possesses a left fraction representation since, by Proposition 3.10, it is an injective homogeneous system. Thus, all our results in the present paper apply to the system Σ_e.

Proposition 4.1 also implies that left fraction representations are unique up to a unimodular transformation, as follows.

Theorem 4.4

Let $\Sigma: S_0(\alpha^m) \rightarrow S_0(R^p)$ be an injective system, and assume it has two left fraction representations $\Sigma = G^{-1}T$ and $\Sigma = G'^{-1}T'$, with factorization spaces $S, S' \subset S_0(R^q)$, respectively. Then, there is a unimodular map $M: S \rightarrow S'$ such that $G' = MG$ and $T' = MT$. Also, the factorization space of any left fraction representation of Σ is S-morphic to $\operatorname{Im} \Sigma$.

Proof

By the injectivity of Σ and our convention that $\operatorname{Im} T = S$ and $\operatorname{Im} T' = S'$, the maps $T: S_0(\alpha^m) \rightarrow S$ and $T': S_0(\alpha^m) \rightarrow S'$ are both set isomorphisms, and whence T^{-1} and T'^{-1} exist. We define $M := T'T^{-1}: S \rightarrow S'$, and we show that M is unimodular. Since M is evidently invertible, we only have to show that $M = T'T^{-1}$ and $M^{-1} = TT'^{-1}$ are both stable. But, this is a direct consequence of the fact that, by Proposition 4.1, the maps T^{-1} and T'^{-1} are both stable maps. Also, clearly $T' = MT$, and, since $G' = T'(T'^{-1}G') = T'(T^{-1}G)$ by the equality $G^{-1}T = G'^{-1}T'$, we have $G' = (T'T^{-1})G = MG$. Finally, from the proof of Theorem 4.2 (stated immediately preceding that theorem), it follows that, every system Σ possessing a left fraction representation, has such a representation with the factorization space $\operatorname{Im} \Sigma$. Combined with the previous part of the present proof, this implies that the factorization space of every left fraction representation of Σ is S-morphic to $\operatorname{Im} \Sigma$. □

As we have repeatedly mentioned, our main motivation in studying left fraction representations of non-linear systems is the need to obtain a simple and transparent characterization of the set of all pairs of stable systems A, B satisfying an equation of the form $AP+BQ=M$, where P, Q and M are fixed stable systems. Considering our discussion in § 1, we are mainly interested in the case where, in the last equation, the systems P and Q arise from a right coprime fraction representation $\Sigma=PQ^{-1}$ of an injective homogeneous system $\Sigma:S_0(\alpha^m)\to S_0(R^p)$. As we have seen in (1.5), left fraction representations of Σ are instrumental in the study of the solutions of this equation. In the next theorem we show that, for injective homogeneous systems $\Sigma:S_0(\alpha^m)\to S_0(R^p)$, the simple procedure outlined in (1.5) yields all solutions of the equation. In this way we obtain a simple and complete parametrization of the set of all solutions of the equation $AP+BQ=M$ discussed in § 1.

Theorem 4.5

Let $\Sigma:S_0(\alpha^m)\to S_0(R^p)$ be an injective homogeneous system, and let $\Sigma=PQ^{-1}$ be a right coprime fraction representation, where $P:S\to \operatorname{Im}\Sigma$ and $Q:S\to S_0(\alpha^m)$, and where $S\subset S_0(R^q)$. Let $\Sigma=G^{-1}T$ be a left fraction representation of Σ, where $G:\operatorname{Im}\Sigma\to S_L$ and $T:S_0(\alpha^m)\to S_L$. Let $M:S\to S$ be any stable map, and let $A:\operatorname{Im}\Sigma\to S_0(R^q)$ and $B:S_0(\alpha^m)\to S_0(R^q)$ be a pair of stable maps satisfying the equation $AP+BQ=M$. Then, a pair of stable maps $A':\operatorname{Im}\Sigma\to S_0(R^q)$ and $B':S_0(\alpha^m)\to S_0(R^q)$ satisfies $A'P+B'Q=M$ if and only if there exists a stable map $h:S_L\to S_0(R^q)$ such that

$$A'=A-hG$$

and

$$B'=B+hT$$

Proof

We have seen in (1.5) that any pair of maps $A':\operatorname{Im}\Sigma\to S_0(R^q)$ and $B':S_0(\alpha^m)\to S_0(R^q)$ of the form $A'=A-hG$ and $B'=B+hT$, where $h:S_L\to S_0(R^q)$ is stable, satisfies $A'P+B'Q=M$. In order to prove the converse direction of our Theorem, let $A':\operatorname{Im}\Sigma\to S_0(R^q)$ and $B':S_0(\alpha^m)\to S_0(R^q)$ be any pair of stable maps satisfying $A'P+B'Q=M$. Then, we have $A'P+B'Q=AP+BQ$, or $(A-A')P=(B-B')Q$. Composing this equation with Q^{-1} on the right, we obtain $(A-A')PQ^{-1}=B'-B$, or $(A-A')\Sigma=B'-B$. Denoting $g:=A-A'$, we have that the map $g:\operatorname{Im}\Sigma\to S_0(R^q)$ is stable, and, by the stability of B and of B', the map $g\Sigma=B'-B:S_0(\alpha^m)\to S_0(R^q)$ is stable as well. Consequently, by Proposition 4.1, the map $h:=gG^{-1}=(g\Sigma)T^{-1}:S_L\to S_0(R^q)$ is stable, and $g=hG$. But then, since $g=A-A'$, we obtain $A-A'=hG$, or $A'=A-hG$. Finally, since $B'-B=(A-A')\Sigma=(hG)(G^{-1}T)=hT$, we also have $B'=B+hT$, and our proof is concluded. □

In conclusion, we have developed in the present paper a complete theory of left and right fraction representations for injective systems $\Sigma:S_0(\alpha^m)\to S_0(R^p)$. The theory is rather simple, possibly even simpler than the linear theory of fraction representations. Fundamentally, the origin of this simplicity is the fact that the systems Σ have the compact domain $S_0(\alpha^m)$. Though the choice of this domain is very natural from the practical point of view, it cannot be adopted in the linear theory, since it would violate linearity. Our discussion was mostly restricted to injective system, but we have seen in § 2 that the injectivity assumption amounts to a transition from the control configuration of Fig. 1 to the control configuration of Fig. 2, and thus is not really restrictive from a control-theoretic point of view. The present theory of fraction representations provides us with a complete parametrization of the set of all pairs of stable systems A, B satisfying the equation $AP+BQ=M$, which, as we saw in § 1, is of central importance to the study of the stabilization of non-linear systems.

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