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Fraction representations of non-linear systems and non-additive state feedback

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The problem of constructing right-coprime fraction representations is considered for non-linear discrete-time systems possessing a continuous realization. It is shown that, given a static, possibly non-additive, state feedback that stabilizes the input/state part of the realization, a right-coprime fraction representation of the entire system can be constructed. The resulting coprime fraction representation has a particularly simple factorization space, and can be used to derive stabilizing controllers for the original system without state access. The construction of appropriate static state feedbacks is described in a companion paper (Hammer 1989 b).

1. Introduction

A right-fraction representation of a non-linear system Σ is a representation of the form $\Sigma = PQ^{-1}$, where P and Q are stable systems. Right-fraction representations play a fundamental role in the theory of stabilization of non-linear systems (see e.g. Hammer 1986). Over the last few years, several methods for the computation of fraction representations of non-linear systems have been presented in the literature (Hammer 1984, Desoer and Kabuli 1988, Sontag 1988). Generally speaking, these methods rely on the theory of additive feedback for non-linear systems. In order to compute a fraction representation using these methods, one has to find an additive feedback compensator that stabilizes the system. Once the stabilizing feedback compensator is known, the desired fraction representation can be computed in a straightforward way. The basic difficulty involved in the use of these methods is the need to find an additive feedback controller that stabilizes the system. As it turns out, in many cases, it is much easier to construct a stabilizing feedback controller that is non-additive (Hammer 1989 b). The purpose of the present note is to show that non-additive feedback controllers can also be employed in the construction of right-coprime fraction representations of non-linear systems. When a right-coprime fraction representation of the system is known, it can be used to devise control configurations that internally stabilize the system, and allow the assignment of desirable dynamical behaviour for the closed loop (Hammer 1987, 1988, 1989 a). The basic methodology presented in this note can be qualitatively outlined as follows.

Consider a non-linear system Σ that can be described in the form

$$\left.\begin{aligned} x_{k+1} &= f(x_k, u_k) \\ y_k &= h(x_k), \quad k = 0, 1, 2, \ldots \end{aligned}\right\} \tag{1.1}$$

where $\{u_k\}_0^\infty$ is the input sequence, consisting of m-dimensional real vectors; $\{y_k\}_0^\infty$ is

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the output sequence, consisting of p-dimensional real vectors; $\{x_k\}_0^\infty$ is an intermediate sequence, consisting of q-dimensional real vectors; and f and h are continuous functions. If a system Σ can be represented in the form (1.1), we say it has a *continuous realization*; if h is the identity function, we call Σ an *input/state* system. Now suppose that the system Σ is described by (1.1), and let Σ_s be the input/state system induced by the recursion $x_{k+1}=f(x_k, u_k)$, $k=0, 1, 2, \ldots$, namely, the input/state part of Σ. Assume that the system Σ_s is enclosed in a closed-loop configuration of the form shown in Fig. 1.

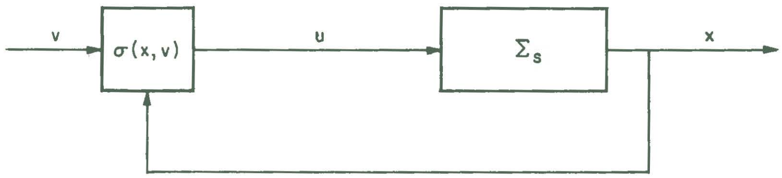

Figure 1.

Here σ is a continuous function through which the feedback loop is closed, and we denote by $\Sigma_{s\sigma}$ the input/output relation induced by the closed loop. Assume now that σ is such that $\Sigma_{s\sigma}$ is stable. The objective of the present paper is to show that, using σ, a right-coprime fraction representation of the original system Σ can be derived. Once such a fraction representation is known, the methods developed by Hammer (1986, 1988, 1989 a) can be used to obtain a control configuration that internally stablizes Σ and allows desirable dynamics assignment, without the need to access the state x. Thus the state feedback function σ is used only as a means of obtaining a fraction representation, and the actual control configuration that stabilizes Σ requires no access to the state. This approach circumvents the need to employ observers.

Our main motivation in writing the present paper derives from some recent results on the stabilization of non-linear systems by static state feedback (Hammer 1989 b). These results indicate that, for a non-linear system, it is quite easy to compute non-additive static state feedback controllers that globally stabilize the system. In the present paper these feedback controllers are used to construct right-coprime fraction representations of non-linear systems possessing continuous realizations.

The first part of the present paper is devoted to the discussion of non-linear input/state systems, and, later, in the last part of § 2, the computation of right-coprime fraction representations for non-linear systems possessing continuous realizations is described. As before, the term *input/state system* refers to a system having a recursive representation of the form

$$x_{k+1}=f(x_k, u_k), \quad k=0, 1, 2, \ldots \tag{1.2}$$

where $x_0, x_1, x_2, \ldots$ is a sequence of p-dimensional real vectors, serving as the output sequence of the system; $u_0, u_1, u_2, \ldots$ is a sequence of m-dimensional real vectors, serving as the input sequence of the system; and $f: \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is a continuous function, called the *recursion function* of the system. The initial condition x_0 of the system has to be specified in order for the recursion to be well defined; however, it can be any vector within the domain over which stabilization is achieved (see Hammer 1989 b). Let Σ denote the system represented by (1.2), and assume Σ is inserted for Σ_s in the closed-loop configuration shown in Fig. 1. In this figure,

$v = \{v_0, v_1, v_2, \ldots\}$ is a sequence of m-dimensional real vectors, serving as the input sequence of the closed loop system, and the feedback function $\sigma: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous. The input/output relation induced by the closed-loop system is denoted by Σ_σ, and a recursive representation for it is given by

$$x_{k+1} = f(x_k, \sigma(x_k, v_k)), \quad k = 0, 1, 2, \ldots \tag{1.3}$$

In particular, if the function σ is of the form

$$\sigma(x, v) = v - \phi(x) \tag{1.4}$$

then the configuration represents additive feedback. However, there is no reason to expect that a non-linear input/state system can be globally stabilized by using static additive feedback, and in fact it is shown by Hammer (1989 b) that static additive feedback is mostly useful for local stabilization. Thus it is necessary to consider non-additive static feedback configurations. When doing so, it is desirable to preserve as many as possible of the fundamental properties of additive feedback. One such property is reversibility, which refers to the fact that an additive feedback operation can be reversed, or 'undone', by another additive feedback operation, applied to the closed loop. To be more specific, consider the closed loop Σ_σ, and assume that an additional feedback loop is closed around it, using the feedback function ω. Denote by $\Sigma_{\sigma\omega}$ the resulting system, and let $f_{\sigma\omega}$ be its recursion function. Then the feedback operation induced by σ is *reversible* if a feedback function ω can be found for which $f_{\sigma\omega} = f$; namely, the feedback operation through ω reverses the feedback operation induced by σ, and restores the original system.

In order to analyse the situation in more detail, the following notation is convenient. For a function $h: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$, $(x, u) \mapsto h(x, u)$, and a fixed element $x \in \mathbb{R}^p$, denote by h_x the function $\mathbb{R}^m \to \mathbb{R}^m$ given by $h_x(u) := h(x, u)$, i.e. the partial function. Letting w be the input of the system $\Sigma_{\sigma\omega}$, direct computation shows that

$$f_{\sigma\omega}(x, w) = f(x, \sigma_x \omega_x(w)) \tag{1.5}$$

(for details see Hammer 1989 b). In order for the feedback operation induced by σ to be reversible, we need

$$f_{\sigma\omega}(x, w) = f(x, w) \tag{1.6}$$

It is then easy to see that the class of continuous feedback functions that induce reversible feedback operations, namely the class of reversible feedback functions, is given by the following definition (Hammer 1989 b).

Definition 1.1

Let Σ be an input/state system with the recursive representation $x_{k+1} = f(x_k, u_k)$. A *reversible feedback function* for Σ is a continuous function $\sigma: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$, $(x, v) \mapsto \sigma(x, v)$ for which the partial function $\sigma_x: \mathbb{R}^m \to \mathbb{R}^m$ is injective for any possible state x.

The class of reversible feedback functions was introduced by Hammer (1989 b) simply as a means to generalize the reversibility property of additive feedback to more general feedback configurations. In the context of our present discussion, reversible feedback functions are of critical importance. As we shall see later, only reversible feedback operations induce right-coprime fraction representations, and thus the sequel depends heavily on the notion of a reversible feedback function.

2. Fraction representations

We start with a description of our basic set-up and notation. As usual, we denote by $\mathbb{R}^m$ the set of all m-dimensional real vectors. For a vector $a=(a^1, a^2, \ldots, a^m) \in \mathbb{R}^m$ let $|a| := \max\{|a^1|, |a^2|, \ldots, |a^m|\}$. By $S(\mathbb{R}^m)$ denote the set of all sequences $u := \{u_0, u_1, u_2, \ldots\}$ of vectors $u_i \in \mathbb{R}^m$, $i = 0, 1, 2, \ldots$. On this space of sequences we define two norms. The first one is the usual l^∞ norm, given, for every element $u \in S(\mathbb{R}^m)$, by $|u| := \sup_{i \geqslant 0}\{|u_i|\}$. The second one is a weighted l^∞ norm ρ, given by $\rho(u) := \sup_{i \geqslant 0}(2^{-i}|u_i|\}$ for all $u \in S(\mathbb{R}^m)$. The topology induced by ρ is our basic underlying topology, and, unless explicitly stated otherwise, all notions of closure, continuity etc. are with respect to this topology. Adopting the input/output approach, a system is regarded simply as a map $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$, transforming input sequences of m-dimensional vectors into output sequences of p-dimensional vectors. For an input sequence $u \in S(\mathbb{R}^m)$ denote by $\Sigma u]_k$ the kth vector of the output sequence Σu, and by $\Sigma u]_i^j$, $j \geqslant i$, the output vectors $\Sigma u]_i, \Sigma u]_{i+1}, \ldots, \Sigma u]_j$.

In order to discuss bounded sequences of vectors, it is convenient to denote by $S(\theta^m)$ the set of all sequences $u \in S(\mathbb{R}^m)$ for which $|u| \leqslant \theta$. Also, given a system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ and a subset $S \subset S(\mathbb{R}^m)$, denote by $\Sigma[S]$ the image of the set S under Σ. Then a system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is *BIBO* (*bounded-input bounded-output*) *stable* if for every real number $\theta > 0$ there is a real number $M > 0$ such that $\Sigma[S(\theta^m)] \subset S(M^p)$. The system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is a *stable* system if it is BIBO stable, and if for every real number $\theta > 0$ the restriction $\Sigma : S(\theta^m) \to S(\mathbb{R}^p)$ is a continuous map with respect to the norm ρ. Let $S_1 \subset S(\mathbb{R}^m)$ and $S_2 \subset S(\mathbb{R}^p)$ be two subsets, and let $M : S_1 \to S_2$ be a system. The system M is *unimodular* if it has a set-theoretic inverse M^{-1}, and if M and M^{-1} are both stable systems.

Of particular importance to our discussion is the class of homogeneous systems, which is defined as follows (Hammer 1987).

Definition 2.1

A system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is a *homogeneous system* if for every real number $\alpha > 0$ and for every subset $S \subset S(\alpha^m)$ the following holds: whenever there is a real number $\theta > 0$ such that $\Sigma[S] \subset S(\theta^p)$, the restriction of Σ to the closure $\bar{S}$ of the set S is a continuous map $\Sigma : \bar{S} \to S(\theta^p)$ (with respect to ρ).

The notion of a homogeneous system is convenient in studies of stability, since, by definition, a homogeneous system is stable (i.e. bounded and continuous) whenever it is BIBO stable. As it turns out, most systems of practical interest are homogeneous systems. In particular, in view of the following statement, which is reproduced from Hammer (1987), all the systems considered in the present paper are homogeneous systems.

Proposition 2.2

A system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ having a continuous realization is a homogeneous system.

The main topic of this paper is the construction of right-coprime fraction representations of non-linear systems, so we now review some of the basic notions of this subject. A *right-fraction representation* of a system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is determined by an integer $q > 0$, a subset $S \subset S(\mathbb{R}^q)$ and a pair of stable systems $P : S \to S(\mathbb{R}^p)$ and

$Q: S \to S(\mathbb{R}^m)$, where Q is a set isomorphism and $\Sigma = PQ^{-1}$. The subset S is then called the *factorization space* of the fraction representation $\Sigma = PQ^{-1}$. Most important for the theory of non-linear control are *coprime* right-fraction representations, which are fraction representations $\Sigma = PQ^{-1}$ in which the stable systems P and Q are right-coprime according to the following definition (Hammer 1987) (for a map $P: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ and a subset $S \subset S(\mathbb{R}^p)$ we denote by $P^*[S]$ the inverse image of the set S under P, namely the set of all input sequences $u \in S(\mathbb{R}^m)$ for which $Pu \in S$).

Definition 2.3

Let $S \subset S(\mathbb{R}^q)$ be a subset. Two stable systems $P: S \to S(\mathbb{R}^p)$ and $Q: S \to S(\mathbb{R}^m)$ are *right-coprime* if the following conditions hold:

(i) for every real number $\tau > 0$ there exists a real number $\theta > 0$ such that

$$P^*[S(\tau P)] \cap Q^*[S(\tau^m)] \subset S(\theta^q)$$

(ii) for every real number $\tau > 0$ the set $S \cap S(\tau^q)$ is a closed subset of $S(\tau^q)$ (with respect to the topology induced by ρ).

An intuitive discussion of the notion of right-coprimeness is given by Hammer (1987). From the control-theoretical point of view, right-coprime fraction representations play a critical role in the construction of compensators that robustly stabilize a non-linear system. Specifically, let $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ be a system, where $\alpha > 0$ is a real number describing the largest amplitude of input sequence the system Σ can accept. Let $\Sigma = PQ^{-1}$ be a right-coprime fraction representation and let $S \subset S(\mathbb{R}^q)$ be its factorization space. As discussed by Hammer (1987), the construction of compensators that stabilize the system Σ involves the computation of two stable systems $A: \operatorname{Im} \Sigma \to S(\mathbb{R}^q)$ and $B: S(\alpha^m) \to S(\mathbb{R}^q)$ satisfying the equation

$$AP + BQ = M \tag{2.1}$$

where $M: S \to S$ is a unimodular system. To review this point, consider the configuration shown in Fig. 2.

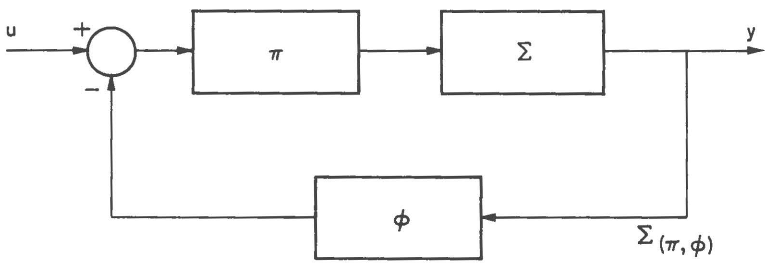

Figure 2.

Here $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is the given system that needs to be controlled; $\pi: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ is a dynamic precompensator and $\phi: S(\mathbb{R}^p) \to S(\mathbb{R}^m)$ is a dynamic feedback compensator, which is connected additively. The closed-loop system is denoted by $\Sigma_{(\pi,\phi)}$. As we have discussed in previous papers, it is particularly convenient to choose the compensators in the form

$$\left.\begin{aligned}\pi &= B^{-1}\\ \phi &= A\end{aligned}\right\} \tag{2.2}$$

where $A: S(\mathbb{R}^p) \to S(\mathbb{R}^m)$ and $B: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ are stable systems, with B being a set isomorphism. Of course, A and B^{-1} have to represent causal systems. For this form of the compensators, the input/output relation induced by the closed-loop system is given by (for details see Hammer 1986)

$$\Sigma_{(\pi,\phi)} = P[AP + BQ]^{-1} \tag{2.3}$$

Now, if the stable systems A and B are chosen so that $AP + BQ = M$, where M is a unimodular system, the input/output relation of the closed loop becomes

$$\Sigma_{(\pi,\phi)} = PM^{-1} \tag{2.4}$$

and it is stable. As discussed in detail by Hammer (1986), the closed-loop system will in fact be internally stable under these circumstances, if the systems A and B satisfy some additional mild assumptions.

The following aspect of (2.4) is of particular interest to us here. In general, the space of input sequences of the closed-loop system $\Sigma_{(\pi,\phi)}$ is required to be of the form $S(\theta^m)$, where $\theta > 0$ is a real number, describing the desired bound on the amplitudes of the input sequences. In view of (2.4), this implies that the domain of M^{-1}, which is the codomain of M, is required to be $S(\theta^m)$. Recalling that S is the factorization space of the coprime fraction representation $\Sigma = PQ^{-1}$ and that $M = AP + BQ$, it follows that $M: S' \to S(\theta^m)$, where S' is an appropriate subset of the factorization space S (for further details see Hammer 1986). Thus we are required to find a subspace S' of the factorization space S that is homeomorphic to $S(\theta^m)$, and to construct the appropriate homeomorphism M. Clearly, a substantial simplification results if the factorization space S is of the form $S(\beta^m)$ for some real $\beta > 0$, since the construction of homeomorphisms $M: S(\beta^m) \to S(\theta^m)$ is a straightforward task.

In the present paper we develop a procedure for the derivation of right-coprime fraction representations $\Sigma = PQ^{-1}$ whose factorization space is of the form $S(\beta^m)$, $\beta > 0$. The critical tool used in this procedure is the theory of static state feedback for non-linear systems developed by Hammer (1989 b). As it turns out, through this theory, right-coprime fraction representations with factorization space $S(\beta^m)$ can be constructed for any non-linear system Σ possessing a continuous realization, provided that the input/state part Σ_s of Σ is stabilizable by the configuration shown in Fig. 1. Necessary and sufficient conditions for the latter are given by Hammer (1989 b). We start with a consideration of non-linear input/state systems.

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be an input/state system with the recursive representation $x_{k+1} = f(x_k, u_k)$, where $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a continuous function, and let $\theta > 0$ be a real number. Assume that there is a reversible feedback function $\sigma: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$ for which the system $\Sigma_\sigma: S(\theta^m) \to S(\mathbb{R}^p)$ of Fig. 1 is stable. Necessary and sufficient conditions for the existence of σ, as well as methods for its construction whenever it exists, are described by Hammer (1989 b). Now, referring to Fig. 1, let $v \in S(\theta^m)$ be an input sequence of the closed-loop system and let $u \in S(\mathbb{R}^m)$ be the corresponding input sequence of the system Σ, so that

$$u = \sigma(\Sigma u, v) \tag{2.5}$$

by which we mean simply that $u_k = \sigma(\Sigma u]_k, v_k)$ for all integers $k \geqslant 0$. In view of the definition of a reversible feedback function, the partial function $\sigma_x:[-\theta,\theta]^m \to \sigma_x[-\theta,\theta]^m$ is a set isomorphism for every state x, and thus has an inverse function $\sigma_x^{-1}:\sigma_x[-\theta,\theta]^m \to [-\theta,\theta]^m$. Denote $\sigma^*(x,u) := \sigma_x^{-1}(u)$. Let S_u denote the set of all sequences $u \in S(\mathbb{R}^m)$ that appear as input sequences of the system Σ in the closed loop shown in Fig. 1 when v varies over $S(\theta^m)$; namely,

$$S_u := \{u \in S(\mathbb{R}^m) : u = \sigma(\Sigma u, v),\ v \in S(\theta^m)\} \tag{2.6}$$

Then we can write

$$v = \sigma^*(\Sigma u, u) \tag{2.7}$$

for all sequences $u \in S_u$. Let $l: S_u \to S(\theta^m)$ be the system given by

$$l(u) := \sigma^*(\Sigma u, u) \tag{2.8}$$

so that $v = l(u)$. In this notation $\Sigma u = \Sigma_\sigma l u$ for all $u \in S_u$, and it follows that the restriction $\Sigma: S_u \to S(\mathbb{R}^p)$ satisfies $\Sigma = \Sigma_\sigma l$. Now assume for a moment that l has a stable inverse $l^{-1}: S(\theta^m) \to S_u$, and denote

$$Q := l^{-1}: S(\theta^m) \to S_u \tag{2.9}$$

Then, recalling that Σ_σ is stable, and setting

$$P := \Sigma_\sigma: S(\theta^m) \to S(\mathbb{R}^p) \tag{2.10}$$

we obtain the right-fraction representation

$$\Sigma = PQ^{-1} \tag{2.11}$$

which is valid over the input space S_u and which has the factorization space $S(\theta^m)$. As it turns out, this fraction representation is in fact coprime, and thus a coprime fraction representation having the factorization space $S(\theta^m)$ is obtained, in line with our basic objective. To complete this discussion, we first show that l is invertible. Recall that a system is *bicausal* if it is invertible and if it and its inverse are both causal systems.

Lemma 2.1

The system $l: S_u \to S(\theta^m)$ of (2.8) is a bicausal isomorphism.

Proof

From the definition of S_u it follows directly that l is surjective. To show that l is also injective, let u and w be two sequences in S_u for which $lu]_0^n = lw]_0^n$ for some integer $n \geqslant 0$. We now show by induction that this implies that $u]_0^n = w]_0^n$. Indeed, $lu]_0 = lw]_0$ means that $\sigma^*(x_0, u_0) = \sigma^*(x_0, w_0)$, where x_0 is the initial condition of Σ. Since σ^* is invertible in its second variable, it follows that $u_0 = w_0$. In preparation for induction, assume that $u_0^i = w_0^i$ for some $i \in \{0, 1, \ldots, n-1\}$. Then, by the strict causality of the system Σ, we have $\Sigma u]_{i+1} = \Sigma w]_{i+1} =: x_{i+1}$. Combining this with the equality $lu]_0^n = lw]_0^n$, which entails $lu]_{i+1} = lw]_{i+1}$, we find $\sigma^*(x_{i+1}, u_{i+1}) = \sigma^*(x_{i+1}, w_{i+1})$. Using again the fact that σ^* is invertible in its second variable, this implies that $u_{i+1} = w_{i+1}$. By induction, we conclude that $u]_0^n = w]_0^n$.

Now, if $lu = lw$ then, by setting $n = \infty$ in the conclusion of the previous paragraph, we find that $u = w$, and l is injective. Since l is also surjective, it follows that $l^{-1}: S(\theta^m) \to S_u$ exists. Moreover, the conclusion of the first paragraph implies directly that

l^{-1} is a causal system. Since l itself is causal by its definition (2.8), we conclude that $l: S_u \to S(\theta^m)$ is a bicausal isomorphism. □

Next, we show that the system Q in the fraction representation (2.11) is stable. (It can in fact be shown that Q is a unimodular system, but we do not need this fact presently. The role of Q can then be interpreted as a transformation of the space S_u, which describes a set of input sequences over which Σ is stable, into the standard input space $S(\theta^m)$.)

Lemma 2.2

The system $Q: S(\theta^m) \to S_u$ is a stable system.

Proof

First note that Q can be represented in the form

$$\left.\begin{array}{l} \begin{bmatrix} w_{k+1} \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} \sigma(f(x_k, w_k), v_{k+1}) \\ f(x_k, w_k) \end{bmatrix}, \quad \begin{bmatrix} w_0 \\ x_0 \end{bmatrix} = \begin{bmatrix} \sigma(x_0, v_0) \\ x_0 \end{bmatrix} \\ u_k = w_k \end{array}\right\} \tag{2.12}$$

$k = 0, 1, 2, \ldots$, where $v \in S(\theta^m)$ is the input sequence of Q, and $u = Qv$ is the output sequence of Q. In view of the continuity of the functions f and σ, this implies that the system Q has a continuous realization, and hence, by Proposition 2.2, it is a homogeneous system. Thus, in order to show that $Q: S(\theta^m) \to S_u$ is stable, we only have to show that there is a real number $\beta > 0$ such that $Q[S(\theta^m)] \subset S(\beta^m)$. Now, since the feedback function σ is such that Σ_σ is stable, there is a real number $\delta > 0$ such that $\Sigma_\sigma[S(\theta^m)] \subset S(\delta^p)$. But then, since x is the output sequence of the closed-loop system Σ_σ, it follows that $x \in S(\delta^p)$. Also, since $u = \sigma(x, v)$, it follows that $u \in \sigma[S(\delta^p) \times S(\theta^m)]$, and, by the continuity of the function $\sigma: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$, it follows that there is a real number $\beta > 0$ such that $u \in S(\beta^m)$. Thus $Q[S(\theta^m)] \subset S(\beta^m)$, and Q is a stable system by Proposition 2.2. □

Finally, we show that the fraction representation (2.11) is in fact right-coprime.

Lemma 2.3

The systems $P: S(\theta^m) \to S(\mathbb{R}^p)$ and $Q: S(\theta^m) \to S_u$ of (2.11) are right-coprime.

Proof

Since the factorization space here is given by $S = S(\theta^m)$, it is clear that $S \cap S(\tau^m)$ is a closed subset (with respect to the topology induced by ρ) for every real number $\tau > 0$. Thus we only have to show that for every real number $\tau > 0$ there exists a real number $\gamma > 0$ such that $P^*[S(\tau^p)] \cap Q^*[S(\tau^m)] \subset S(\gamma^m)$. However, in our case this inclusion is a direct consequence of the fact that the factorization space S, i.e. the domain of P and Q, is simply $S = S(\theta^m)$, which implies that the inclusion is valid for $\beta = \theta$. □

Thus, using the stabilizing reversible feedback function σ, we have constructed a right-coprime fraction representation of the system Σ. The main advantage of this fraction representation over the fraction representations derived by Hammer (1987) is that the present fraction representation has the factorization space $S(\theta^m)$. As discussed earlier in this section, the latter is highly instrumental in the construction of compensators that yield stabilization and desired dynamics assignment for the closed-loop system illustrated in Fig. 2. It is also important to note that the numerator P and the denominator Q of our fraction representation are both implementable systems (by (2.10), (1.3) and (2.12)), and that Q is bicausal (by Lemma 2.1 and (2.9)).

We turn now to the computation of right-coprime fraction representations for systems possessing continuous realizations. The method we use is a direct application of the ideas developed so far in this section. Consider a non-linear system $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ having the continuous realization (1.1), where $f: \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^q$ and $h: \mathbb{R}^q \to \mathbb{R}^p$ are continuous functions, and where $u \in S(\mathbb{R}^m)$ is the input sequence of Σ, $y = \Sigma u$ is the output sequence and $x \in S(\mathbb{R}^q)$ is an intermediate sequence. Then the system $\Sigma_s: S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ given by the recursion $x_{k+1} = f(x_k, u_k)$, $k = 0, 1, 2, \ldots$, is an input/state system, serving as the input/state part of Σ. Assume that there is a reversible feedback function $\sigma: \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^m$ for which the closed-loop system $\Sigma_{s\sigma}: S(\theta^m) \to S(\mathbb{R}^q)$ is stable. Using the function σ, we can construct a right-coprime fraction representation for the input/state system Σ_s, as in (2.11). Let $\Sigma_s = P_s Q^{-1}$ be the resulting fraction representation, and note that its factorization space is given by $S(\theta^m)$. Specifically,

$$\left.\begin{aligned} P_s = \Sigma_{s\sigma}: S(\theta^m) \to S(\mathbb{R}^q), \quad & Q = l^{-1}: S(\theta^m) \to S_u \\ S_u := \{u \in S(\mathbb{R}^m): u = \sigma(\Sigma_s u, v),\ & v \in S(\theta^m)\} \\ l: S_u \to S(\theta^m): \quad l(u) := \sigma^*(\Sigma u, u) \quad & \text{for all } u \in S_u \end{aligned}\right\} \tag{2.13}$$

But then, by the continuity of the function h, it follows that the system

$$P := hP_s: S(\theta^m) \to S(\mathbb{R}^p) \tag{2.14}$$

given, for all $v \in S(\theta^m)$, by $Pv]_k = h(P_s v]_k)$, $k = 0, 1, 2, \ldots$, is a stable system, and

$$\Sigma = PQ^{-1} \tag{2.15}$$

Thus we have obtained a right-fraction representation of the system Σ. A continuous realization of the system P is given by

$$\left.\begin{aligned} x_{k+1} &= f(x_k, \sigma(x_k, v_k)) \\ y_k &= h(x_k) \end{aligned}\right\} \tag{2.16}$$

where $v \in S(\theta^m)$ and $y = Pv$. A continuous realization of Q is described by (2.12). An argument very similar to that used in the proof of Lemma (2.3) further shows that the fraction representation (2.15) is in fact right-coprime, and we obtain the following.

Theorem 2.1

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a system having a continuous realization of the form (1.1) and let $\Sigma_s: S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ be the input/state system induced by the recursion $x_{k+1} = f(x_k, u_k)$, where f is from (1.1). Let $\sigma: \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^m$ be a reversible feedback function that stabilizes the system Σ_s. Then the fraction representation $\Sigma = PQ^{-1}: S_u \to S(\mathbb{R}^p)$ of (2.15) is right-coprime and has the factorization space $S(\theta^m)$. Furthermore, the systems $P: S(\theta^m) \to S(\mathbb{R}^p)$ and $Q: S(\theta^m) \to S_u$ both possess continuous realizations, and Q is bicausal.

Note that the computation of the space S_u is of no importance here. From the control-theoretic point of view, only the factorization space of the coprime fraction representation is of importance, as we have discussed earlier in this section. The space S_u, which forms the input space of the system Σ within the closed loop, is automatically generated by the closed-loop system. As can be seen from (2.13), the space S_u depends on the state feedback function σ, and may vary from one fraction representation to another. To conclude, once a *reversible* stabilizing feedback function σ for the input/state part Σ_s of Σ is known, a right-coprime fraction representation of the entire system Σ can be directly computed. For some explicit examples of the computation of stabilizing reversible feedback functions see Hammer (1989 b).

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