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FRACTION REPRESENTATIONS AND ROBUST STABILIZATION OF NONLINEAR SYSTEMS

by

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ABSTRACT

The purpose of this note is to provide a survey of the theory of fraction representation and robust stabilization of nonlinear systems developed by the author over the last few years. The note contains an exposition of the main results obtained so far and some examples, but no proofs are included. The results are all explicit and implementable.

INTRODUCTION

Over the last few years, the author has been engaged in the development of a theory of stabilization for nonlinear systems (HAMMER [1984a,b, 1985a,b, 1986, 1987a,b, and 1988]). The basic mathematical notion on which this theory rests is the notion of fraction representations of nonlinear systems. Generally speaking, a fraction representation of a nonlinear system is a factorization of the system into a composition of two nonlinear systems, one of which is stable and the other is the inverse of a stable system. More specifically, one distinguishes between two kinds of fraction representations - a right fraction representation and a left fraction representation. A right fraction representation of a nonlinear system Σ is a representation of the form $\Sigma = PQ^{-1}$, where P and Q are stable systems, with Q being invertible (i.e., a set isomorphism). A left fraction representation of the system Σ is of the form $\Sigma = G^{-1}T$, where G and T are stable systems, with G being invertible. As it turns out, and as we manifest throughout the present note, fraction representations play a fundamental role in the theory of stabilization for nonlinear systems, and their construction is instrumental for the computation of compensators that stabilize a given system.

The general appearance of the stabilization theory we develop resembles very closely the transfer matrix theory of linear systems. The mathematical techniques we use for the nonlinear case are, of course, of a totally different nature, and no transforms are involved. In our presentation, we limit ourselves to the use of common mathematical techniques, and the general mathematical background we use can be found in any basic book on topology (e.g., KURATOWSKI [1961]). We shall discuss the robust stabilization of discrete-time nonlinear recursive systems, and the results we obtain are all explicit and can be directly implemented on digital computers. The purpose of this note is to provide a brief survey of the status of our theory at the present time. For proofs and detailed technical discussions of the results we survey here, see the appropriate full text papers.

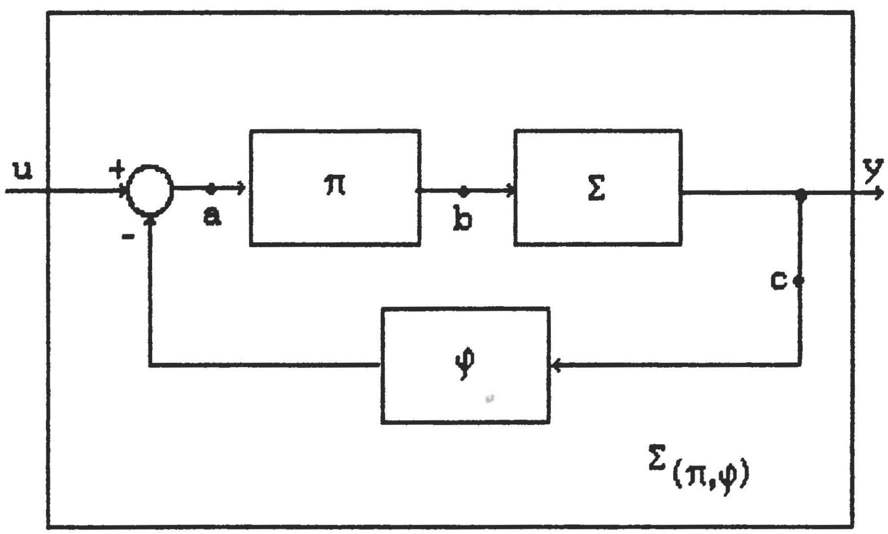
We mention briefly the literature background. As we said, the results surveyed in this note are taken from HAMMER [1984a,b, 1985a,b, 1986, 1987a,b, and 1988].

Alternative recent studies on the stabilization of nonlinear systems can be found in VIDYASAGAR [1980], SONTAG [1981], DESOER and LIN [1984], ISIDORI [1985], the references cited in these papers, and others. Studies on the effect of feedback on system uncertainties appeared in BLACK [1934], BODE [1945], NEWTON, GOULD, and KAISER [1957], ZAMES [1966 and 1981], ROSENBROCK [1970 and 1974], DESOER and VIDYASAGAR [1975], KIMURA [1984], the references cited in these papers, and others.

2. MOTIVATION AND GENERALITIES

The basic control configuration that we use in our study of the stabilization of nonlinear systems is the following classical one.

(2.1)



Here, Σ is the given system which needs to be stabilized, π is a dynamic precompensator, φ is a dynamic feedback compensator, and $\Sigma_{(\pi,\varphi)}$ denotes the closed loop system. We have repeatedly concluded in our studies of the nonlinear stabilization problem that it is of particular advantage to choose the precompensator π and the feedback compensator φ in the form

$$\pi = B^{-1}, \qquad (2.2)$$
$$\varphi = A,$$

where A and B are stable systems with B being invertible, and where A and B^{-1} are causal. The advantage of using this particular form of the compensators is twofold. First, this form of the compensators leads to particularly simple and transparent conditions for input/output stabilization, as we show in a moment. Second, the conditions for internal stability become substantially simplified when this form of compensators is used, to the point where internal stability is almost implied by input/output stability, and an explicit derivation of compensators that internally stabilize the system becomes possible. The way these advantages come about will become clear from our ensuing discussion. It is not less important to note that the price that we pay for restricting ourselves to compensators of the form (2.2) is rather low. With this choice of compensators we can achieve virtually arbitrary dynamics assignment for the internally stable closed loop (HAMMER [1987b]), and we can design the closed loop to be robustly stable (HAMMER [1988]). Thus, this configuration allows us to do more or less everything we would like to do from a stabilization point of view, with a minimal amount of complication.

Throughout our discussion, we make the basic assumption that the amplitudes of the input sequences to any of the systems we consider are bounded by a fixed bound, namely, that there is a real number $\alpha > 0$ such that all the input sequences to our systems are of amplitude not exceeding α. In most practical situations, this does not really amount to an assumption, but rather to a description of the actual physical reality. The input sequences, being generated by a physical device, are naturally of bounded amplitudes, the bound being determined, for instance, by saturation phenomena.

Let us now turn to a preliminary analysis of the control cofiguration (2.1). Assume that the system Σ has a right fraction representation $\Sigma = PQ^{-1}$, and that the compensators π and φ are given by (2.2). Then, it can be readily seen (e.g., HAMMER [1984a]) that, under some standard mild assumptions, the input/output relationship induced by the closed loop system $\Sigma_{(\pi,\varphi)}$ is given by

(2.3) $$\Sigma_{(\pi,\varphi)} = \Sigma\pi[I + \varphi\Sigma\pi]^{-1} = PQ^{-1}B^{-1}[I + APQ^{-1}B^{-1}]^{-1} = P[AP + BQ]^{-1}.$$

Denoting

(2.4) $$M := AP + BQ,$$

we obtain that

(2.5) $$\Sigma_{(\pi,\varphi)} = PM^{-1},$$

in close analogy to the linear situation. Clearly, if the stable systems A and B are selected so that the stable system M also has a stable inverse M^{-1}, then the closed-loop system $\Sigma_{(\pi,\varphi)}$ becomes input/output stable. In fact, as we discuss later, $\Sigma_{(\pi,\varphi)}$ will be internally stable under these circumstances whenever the systems A and B satisfy some additional mild requirements (HAMMER [1986b and 1987b]). A stable system M which is invertible and whose inverse M^{-1} is also stable is called a unimodular system. For the existence of stable systems A, B satisfying the equation $AP + BQ = M$ with M unimodular, we need P and Q to be right comprime, as we elaborate in a later section.

It is rather obvious from our discussion so far that the problem of finding stable systems A and B which satisfy the equation $AP + BQ = M$, where P, Q, and M are given, is of central importance to our discussion. In order to be able to choose the compensators π and φ most convenient for implementation, we would in fact like to know all pairs of stable systems A, B satisfying that equation. It is comforting to know that in order to obtain all such pairs of stable systems, all we need is one pair, and, given one pair, all other pairs can be obtained in a straightforward way from transparent parametrization equations. For this purpose we need left fraction representations of nonlinear systems.

Let $\Sigma = G^{-1}T$, where G and T are stable systems, be a left fraction representation of the given system Σ. Recalling the right fraction representation $\Sigma = PQ^{-1}$ from before, we obtain $G^{-1}T = PQ^{-1}$, or

$$TQ = GP.$$

Now, assume we have one pair of stable systems A, B satisfying $AP + BQ = M$. To obtain other pairs of such systems, we can proceed simply as follows. Choose an arbitrary stable system h, and define the stable systems

$$\begin{aligned} A' &:= A - hG, \\ B' &:= B + hT. \end{aligned} \tag{2.6}$$

Then, using the fact that $GP = TQ$, we obtain

$$A'P + B'Q = AP - hGP + BQ + hTQ = AP + BQ = M,$$

and A', B' satisfy our equation. Thus, for every choice of h we obtain a new pair of solutions, and we see that left fraction representations allow us to parametrize solutions of our basic equation in a rather transparent way. In fact, for the systems we consider in this note, (2.6) provides all pairs of solutions A', B' of the equation $A'P + B'Q = M$, when one such pair A, B is known. We conclude that left fraction representations also are of crucial importance to the theory of stabilization of nonlinear systems.

Returning now to equation (2.5), we see that the unimodular transformation M controls the dynamical properties of the closed loop system $\Sigma_{(\pi,\varphi)}$. By choosing M appropriatly, we can achieve dynamics assignment for our systems. Of course, detailed attention has to be given to the problem of internal stability of the closed loop, and we shall describe in a later section how internal stability of the configuration can be guarantied.

The final topic we would like to review in this note is the question of robust stabilization of nonlinear systems. Suppose the accurate description of the system Σ that needs to be stabilized is not known, and that only a nominal description of Σ is given. We denote the nominal description of the system that has to be stabilized by Σ_n, and we allow the actual system Σ to deviate from its nominal description. The central question in the theory of robust stabilization is the following. Is it possible to use the nominal description Σ_n to design the control configuration (2.1) in such a way that it will preserve its stability when the actual system Σ is inserted in it, instead of the nominal system Σ_n for which it was designed. We will describe in a later section a solution to the robust stabilization problem obtained in HAMMER [1988]. The underlying ideas on which this solution is based can be qualitatively (and quite inaccurately) described as follows. Let $\Sigma = PQ^{-1}$ be a fraction representation of the given system. Suppose we have one approppriate pair of systems A and B for which $M := AP + BQ$ is unimodular. The systems P and Q, which arise from a right fraction representation of the system Σ, depend, of course, on Σ. Consequently, deviations of Σ from its nominal value Σ_n will cause deviations of P and of Q from their nominal values. Let $\Sigma_n = P_nQ_n^{-1}$ be a fraction representation of the nominal system. Let Σ be the actual system with the deviation, and suppose we can construct for it a fraction representation $\Sigma = PQ^{-1}$ in which the numerator P satisfies $P = P_n$, where P_n is the numerator of the fraction representation of the nominal system Σ_n. Namely, assume that the effect of the deviation can be completely described by a deviation of the denominator system Q from its nominal value Q_n. Denote $\omega := Q - Q_n$, and notice that ω is a stable system, and that $Q = Q_n + \omega$. Suppose further that, for every real number $\varepsilon > 0$, there is a causal and stable system A_ε satisfying the equation $A_\varepsilon P_n + \varepsilon Q_n = M$. Notice that when the latter holds and (2.2) is used, the system Σ can be stabilized using $B = \varepsilon I$ (and $A = A_\varepsilon$), in which case the precompensator $\pi = (1/\varepsilon)I$ is a simple amplifier. By taking ε arbitrarily small, we can arbitrarily increase the gain of this amplifier, and thus arbitrarily increase the forward path gain. Finally, suppose there is a real number $\delta > 0$ such that the system $M' := M + \mathfrak{M}$ stays unimodular for every stable system $\mathfrak{M}$ with 'magnitude' not exceeding δ, so that a deviation of ' less than δ ' does not destroy the unimodularity of M. The existence of δ as well as its value depend, of course, on the nature of the particular unimodular system M.

Now, assume that the nominal system Σ_n is stabilized using the compensators induced by $A = A_\varepsilon$ and $B = \varepsilon I$, via (2.2). Then, when the system Σ is inserted in the loop instead of the nominal system Σ_n for which the loop was designed, we obtain, recalling the fraction representation $\Sigma = PQ^{-1}$, that

$$AP + BQ = A_\varepsilon P_n + \varepsilon(Q_n + \omega) = A_\varepsilon P_n + \varepsilon Q_n + \varepsilon\omega = M + \varepsilon\omega =: M'.$$

Whence, if we choose ε small enough so that the 'magnitude' of $\mathfrak{M} := \varepsilon\omega$ is smaller than δ, the system M' will still be unimodular, and the input/output relationship $\Sigma_{(\pi,\varphi)} = PM'^{-1}$ induced by the closed loop will remain stable, despite the deviation in the system Σ. Thus, the deviation will not destroy stability. Basically, our discussion in this paragraph is but a restatement of the qualitative principle that, in a closed feedback loop, high gain in the forward path can counteract deviations in the parameters of the forward path systems, a principle which has been widely accepted on an intuitive level ever since the classical work of BLACK [1934] on linear feedback systems. The main advantage of the particular form in which we formulate this principle here is that, in this formulation, the principle can be readily applied to nonlinear situations, and incorporated within the requirements of the theory of internal stability.

The qualitative ideas that we presented in this section form the crude material for the theory of fraction representations and robust stabilization that we survey in the remaining parts of this note. As we shall see, the theory is rather general in its scope, and the results it provides are explicit and implementable.

3. THE BASIC FRAMEWORK AND FRACTION REPRESENTATIONS

The systems we consider are discrete-time systems, accepting sequences of m-dimensional real vectors as their input, and generating sequences of p-dimensional real vectors as their output. To introduce our notation, we let R be the set of real numbers, and, for an integer $m > 0$, we let R^m be the set of all m-dimensional real vectors. By R^0 we simply mean the zero element 0. We denote by $S(R^m)$ the set of all sequences of the form $u_0, u_1, u_2, \ldots,$ with each element u_i belonging to R^m. Given a sequence $u \in S(R^m)$ and an integer $i \geq 0$, we denote by u_i the i-th element of the sequence, and we interpret the integer i as the time marker. For two integers $i > j \geq 0$, we denote by u_j^i the set of elements $u_j, u_{j+1}, \ldots, u_i$. In the set $S(R^m)$ we induce the usual operation of addition elementwise, so that, given a pair of sequences $u, v \in S(R^m)$, the sum $w := u + v$ is again a sequence in $S(R^m)$, with each one of its elements being given by $w_i := u_i + v_i$, $i = 0,1,2, \ldots$.

Adopting the input/output point of view, we conceive a system as a device that transforms input sequences into output sequences. In accurate terms, a system Σ is simply a map $\Sigma : S(R^m) \to S(R^p)$, transforming input sequences from $S(R^m)$ into output sequences from $S(R^p)$, where m and p are arbitrary positive integers. In order to be able to obtain results which are simple, explicit, computable, and implementable, we shall not discuss here systems on this level of generality. Instead, we shall restrict ourselves to recursive systems which have their state as output, namely, to systems Σ possessing a recursive representation of the form

$$(3.1) \qquad x_{k+1} = f(x_k, u_k),$$

where $f : R^p \times R^m \to R^p$ is a function, which we shall usually assume to be continuous. Here, the input sequence of the system is $\{u_k\}$ and its output sequence is $\{x_k\}$, and we assume that the initial condition x_0 is specified. The function f is called a recursion function of the system Σ. Given a subspace $S \subset S(R^m)$, we denote by $\Sigma[S]$ the image of the set S through Σ, namely, the set of all output sequences that Σ generates from input sequences belonging to S. Also, given a subspace $S' \subset S(R^p)$, we denote by $\Sigma^*[S']$ the inverse image of the set S' through Σ, namely, the set of all input sequences that generate output sequences belonging to the set S'.

We review now briefly some notions related to causality of systems. A system $\Sigma : S(R^m) \to S(R^p)$ is causal (respectively, strictly causal) if the following holds for every pair of input sequences $u, v \in S(R^m)$: for all integers $i \geq 0$ for which $u_0^i = v_0^i$, one also has $\Sigma u]_0^i = \Sigma v]_0^i$ (respectively, $\Sigma u]_0^{i+1} = \Sigma v]_0^{i+1}$). A system $M : S(R^m) \to S(R^m)$ is a bicausal system if it is invertible, and if M and M^{-1} are both causal systems.

Most of our discussion is related, of course, to the stability of systems. The notion of stability that we adopt is in the spirit of the Lyapunov notion of stability, and thus is related to the continuity of the system as a map. For the purpose of introducing the notion of stability, we need to induce some norms on the space of sequences $S(R^m)$.

First, let $w = (w^1, ..., w^m)$ be a vector in R^m. We denote $|w| := \max \{|w^i|, i = 1, ..., m\}$, the maximal absolute value of the coordinates of w. Next, we define a norm on the space $S(R^m)$ given, for any element $u \in S(R^m)$, by $\rho(u) := \sup \{2^{-i}|u_i|, i = 0,1,2, ... \}$, and we note that this is simply a weighted ℓ^∞-norm. We use this norm to define a metric $\rho(u,v)$ on $S(R^m)$, by letting $\rho(u,v) := \rho(u-v)$ for every pair of elements $u, v \in S(R^m)$. Whenever referring to continuity, we shall always mean continuity with respect to the topology induced by the metric ρ, unless explicitly stated otherwise. It will also be convenient for us to use the notation $|u| := \sup \{|u_i|, i = 0,1,2, ...\}$ for an element $u \in S(R^m)$. Then, for a real number $\theta > 0$, we denote by $S(\theta^m)$ the set of all elements $u \in S(R^m)$ satisfying $|u| \le \theta$, namely, the set of all sequences bounded by θ. A system $\Sigma : S(R^m) \to S(R^p)$ is BIBO (Bounded-Input Bounded-Output)-stable if, for every real number $\theta > 0$, there is a real number $N > 0$ such that $\Sigma[S(\theta^m)] \subset S(N^p)$. Finally, we say that a system $\Sigma : S(R^m) \to S(R^p)$ is stable if it is BIBO-stable, and if, for every real number $\theta > 0$, the restriction $\Sigma : S(\theta^m) \to S(R^p)$ is a continuous map.

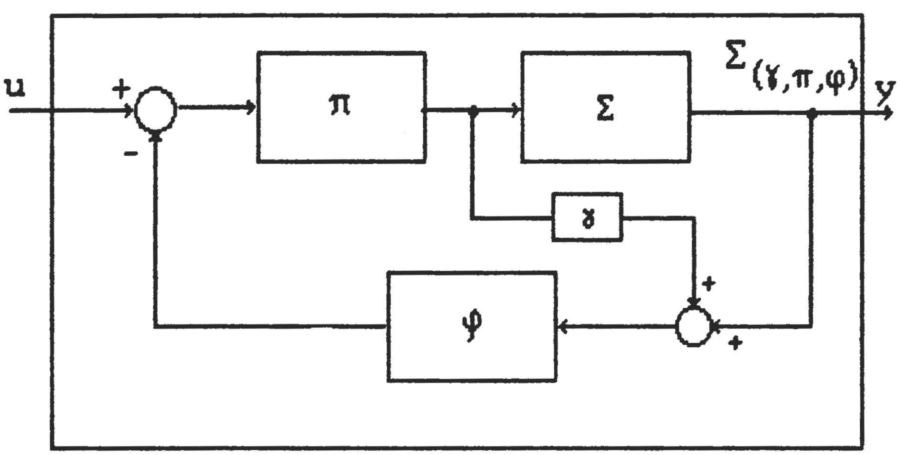
Before turning to a review of our theory of fraction representations for nonlinear systems, we wish to discuss two basic assumptions that we make in the development of our framework. The first assumption is that all the systems we consider are operated by bounded input sequences, namely, that there is a fixed real number $\alpha > 0$ such that all our systems have $S(\alpha^m)$ as their domain. As we have remarked already in an earlier section, this is hardly a restrictive assumption from the practical point of view. In practice, input sequences are generated by a physical device, and their maximal amplitude is limited by the physical characteristics of that device. The second assumption we make is that the system Σ that needs to be stabilized is an injective (one to one) map. At first glance, this looks like a restrictive assumption, since many systems of practical interest are, of course, not injective systems. However, further reflection shows that the assumption that the system that needs to be stabilized is an injective system is not really restrictive, for the following reason. Assume that the system Σ that needs to be stabilized is a strictly causal system. This is always true for systems having recursive representations of the form (3.1). Then, instead of stabilizing the system Σ directly, consider the stabilization of the system $I+\Sigma$, the sum of Σ and the identity system I, ignoring for a second the fact that this sum might not be well defined due to different input space and output space dimensionalities. Then, the strict causality of Σ implies that $I+\Sigma$ is bicausal, and hence injective. Moreover, if we stabilize the system $I+\Sigma$, we shall also obtain stabilization of the original system Σ (in a somewhat different control configuration), as we now show.

Let $\Sigma : S(R^m) \to S(R^q)$ be a strictly causal system. Let $p := \max\{m, q\}$, and define the identity injection maps $\mathcal{J}_1 : S(R^m) \to S(R^p)$ and $\mathcal{J}_2 : S(R^q) \to S(R^p)$ as follows. If $q \geq m$, write $S(R^p) = S(R^q) = S(R^m) \times S(R^{q-m})$, let $\mathcal{J}_1 : S(R^m) \to S(R^p) : \mathcal{J}_1[S(R^m)] = S(R^m) \times 0$ be the obvious identity injection, and let $\mathcal{J}_2 : S(R^q) \to S(R^p)$ ($=S(R^q)$) be the identity map. If $q < m$, write $S(R^p) = S(R^m) = S(R^q) \times S(R^{m-q})$, let $\mathcal{J}_2 : S(R^q) \to S(R^p) : \mathcal{J}_2[S(R^q)] = S(R^q) \times 0$ be the obvious identity injection, and let $\mathcal{J}_1 : S(R^m) \to S(R^p)$ ($=S(R^m)$) be the identity map. Then, as we show in a minute, the system

(3.2) $$\Sigma_\gamma := \gamma\mathcal{J}_1 + \mathcal{J}_2\Sigma \ : \ S(R^m) \to S(R^p),$$

where γ is a $p \times p$ constant nonsingular matrix, is injective by the strict causality of the system Σ. The implementation of the injections $\mathcal{J}_1$ and $\mathcal{J}_2$ is very simple - it just amounts to increasing the dimension of some vectors through augmentation by entries of zeros (see HAMMER [1987b] for details). To simplify our notation, we shall usually abbreviate and denote $\mathcal{J}_1 u$ by u and $\mathcal{J}_2 y$ by y. It can be seen that, when stabilizing the system Σ_γ in the configuration (2.1), we in fact obtain stabilization of the original system Σ in the following configuration.

(3.3)



(Note that in the configuration (3.2), γ is to be interpreted as $\gamma\mathcal{J}_1$, in consistency with our notational convention.)

Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system. Then, the system Σ_γ of (3.2) is an injective system whenever the pxp matrix γ is nonsingular, and thus Σ_γ possesses a left inverse. Moreover, when the original system Σ is recursive, the left inverse of Σ_γ is very easy to compute. Indeed, assume that Σ has a recursive representation $x_{k+1} = f(x_k, u_k)$. Let $u \in S(R^m)$ be an input sequence, and let $x := \Sigma u$ be the corresponding output sequence. Denoting $z := \Sigma_\gamma u$, and using the abbreviated notation mentioned in the previous paragraph, we obtain $z = x + \gamma u$, so that $z_i = x_i + \gamma u_i$ for all integers $i \geq 0$. Therefore, $z_{k+1} = x_{k+1} + \gamma u_{k+1} = f(x_k, u_k) + \gamma u_{k+1} = f((z-\gamma u)_k, u_k) + \gamma u_{k+1}$, and, invoking the invertibility of γ, we obtain

$$u_{k+1} = \gamma^{-1}\{z_{k+1} - f((z-\gamma u)_k, u_k)\}\ ,\ k = 0,1,2,\ \ldots,$$

(3.4)

$$u_0 = \gamma^{-1}\{z_0 - x_0\},$$

where x_0 is the given initial condition of the system Σ, and where the relations are valid for any sequence $z \in \operatorname{Im} \Sigma_\gamma$. Thus, the input sequence u of Σ_γ can be readily computed from the output sequence z of Σ_γ in a recursive manner, using the given recursion function and initial conditions of the system Σ. This evidently amounts to a left inversion of the system Σ_γ, and we shall use these formulas repeatedly in the sequel. It is also clear from (3.4) that this left inverse is causal, and we have the following

(3.5) PROPOSITION. Let $\Sigma : S(R^m) \to S(R^q)$ be a strictly causal recursive system having a recursive representation $x_{k+1} = f(x_k, u_k)$. Let $p := \max\{m, q\}$, and let γ be a pxp constant invertible matrix. Then, the system $\Sigma_\gamma : S(R^m) \to \operatorname{Im} \Sigma_\gamma$ defined by (3.2) is a bicausal system.

We can summarize our discussion in the last few paragraphs by saying that we can always transform our situation into one where the system that needs to be stabilized is injective, even if the original system Σ is not injective. Consequently, from a stabilization point of view, it is not overly restrictive to limit our attention to the discussion of injective systems.

We provide now a brief survey of the theory of right and of left fraction representations for an injective system $\Sigma : S(\alpha^m) \to S(R^p)$, where $\alpha > 0$ is a fixed, but otherwise arbitrary, real number. As we shall see, the theory is surprisingly simple.

A right fraction representation of a system $\Sigma : S(\alpha^m) \to S(R^p)$ involves an integer $q > 0$, a subspace $S \subset S(R^q)$, called the factorization space, and a pair of stable systems $P : S \to S(R^p)$ and $Q : S \to S(\alpha^m)$, where Q is invertible, so that $\Sigma = PQ^{-1}$. Of particular importance to us are coprime right fraction representations, which are fraction representations in which the systems P and Q are right coprime according to the following definition (HAMMER [1985a, 1987a]).

(3.6) DEFINITION. Let $S \subset S(R^q)$ be a subspace. A pair of stable systems $P : S \to S(R^p)$ and $Q : S \to S(R^m)$ are right coprime if the following two conditions are satisfied.

(i) For every real $\tau > 0$ there is a real $\theta > 0$ such that

$$P^*[S(\tau^p)] \cap Q^*[S(\tau^m)] \subset S(\theta^q).$$

(ii) For every real $\tau > 0$, the set $S \cap S(\tau^q)$ is a closed subset of $S(\tau^q)$.□

It is quite easy to see why right coprime fraction representations are important to our discussion. In (2.4) we saw that the solution of the stabilization problem involves the search for a pair of stable systems A and B satisfying the equation $AP + BQ = M$, where P and Q arise from a fraction representation $\Sigma = PQ^{-1}$ of the given system Σ, and where M is a unimodular system. The existence of such systems A and B is guarantied whenever P and Q are right coprime, as follows (HAMMER [1987a]).

(3.7) THEOREM. Let $\Sigma : S(\alpha^m) \to S(R^p)$ be an injective system, and assume it has a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P : S \to S(R^p)$ and $Q : S \to S(\alpha^m)$, and where $S \subset S(R^q)$ for some integer $q > 0$. Then, for every unimodular system $M : S \to S$, there exists a pair of stable systems $A : S(R^p) \to S(R^q)$ and $B : S(\alpha^m) \to S(R^q)$ such that $AP + BQ = M$.

Theorem (3.7) underscores the importance of right coprime fraction representations to our discussion. At the same time, it opens a new question - what systems possess right coprime fraction representations. The existence of right coprime fraction representations is related in a fundamental way to the concept of a homogeneous system, which is defined as follows (HAMMER [1985a, 1987a]).

(3.8) DEFINITION. A system $\Sigma : S(R^m) \to S(R^p)$ is a homogeneous system if the following holds for every real number $\alpha > 0$: for every subspace $S \subset S(\alpha^m)$ for which there exists a real number $\tau > 0$ satisfying $\Sigma[S] \subset S(\tau^p)$, the restriction of Σ to the closure $\bar{S}$ of S in $S(\alpha^m)$ is a continuous map $\Sigma : \bar{S} \to S(\tau^p)$.□

As the next statement shows, (injective) homogeneous systems possess right coprime fraction representations, and they are the only systems possessing such representations. Thus, the concept of a homogeneous system provides a complete characterization of the existence of right coprime fraction representations, in terms of input/output properties of the system (HAMMER [1985a, 1987a]).

(3.9) THEOREM. An injective system $\Sigma : S(\alpha^m) \to S(R^p)$ has a right coprime fraction representation if and only if it is a homogeneous system.

Of course, the obvious question now is - how common are homogeneous systems in practical applications. A partial answer to this question is given by the following statement, which shows that all the systems we consider in our present note are homogeneous. More general classes of homogeneous systems are described in the references (HAMMER [1987a]).

(3.10) PROPOSITION. Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system. If Σ has a recursive representation $x_{k+1} = f(x_k, u_k)$ with a continuous recursion function f, then Σ is a homogeneous system.

As we have discussed in detail earlier in this section, we usually prefer to study the stabilization of the system Σ_γ of (3.2) instead of studying directly the stabilization of the given system Σ. The reasons for this are twofold. First, the theory of fraction representations for the case of injective systems is simpler and more transparent in its appearance, and Σ_γ is always injective when Σ is strictly causal. Secondly, the solution to the stabilization problem becomes simpler if the system Σ_γ is used instead of Σ, even in the case where Σ is injective, due to the simplicity of the inversion formulas (3.4) for Σ_γ. It is therefore of interest to know that when the system Σ is homogeneous, so also is the system Σ_γ (HAMMER [1987a]).

(3.11) PROPOSITION. Let $\Sigma : S(R^m) \to S(R^p)$ be a homogeneous system, and let Σ_γ be defined as in (3.2). Then, $\Sigma_\gamma : S(R^m) \to S(R^p)$ is a homogeneous system.

It is quite easy to construct a right coprime fraction representation for an injective homogeneous system $\Sigma : S(\alpha^m) \to S(R^p)$. Indeed, since Σ is injective, its restriction $\Sigma : S(\alpha^m) \to \mathrm{Im}\,\Sigma$ is a set isomorphism, and, consequently, it possesses an inverse $\Sigma^{-1} : \mathrm{Im}\,\Sigma \to S(\alpha^m)$. We have shown in HAMMER [1987a, section 3] that Σ^{-1} is a stable system. This is a significant departure from the situation in the case of linear systems, where only very special systems possess stable inverses. The root of this departure is the fact that the domain $S(\alpha^m)$ of our systems here is compact in our topology, a fact that originates from the realistic assumption that all systems are operated by bounded input sequences. Thus, we see that the nonlinear framework allows us to take advantage of inherent restrictions in the physical operation of practical systems to simplify the mathematical structure of the problem. Defining the systems

$$(3.12)\qquad \begin{aligned} P &:= I : \mathrm{Im}\,\Sigma \to \mathrm{Im}\,\Sigma, \\ Q &:= \Sigma^{-1} : \mathrm{Im}\,\Sigma \to S(\alpha^m), \end{aligned}$$

where I denotes the identity system, we obtain a right fraction representation $\Sigma = PQ^{-1}$, which, as one can readily see, is right coprime. Once we have one right coprime fraction representation $\Sigma = PQ^{-1}$ of the system Σ, any other right coprime fraction representation of Σ is of the form $\Sigma = P_1Q_1^{-1}$ where $P_1 = PM$ and $Q_1 = QM$, and where M is a unimodular system (HAMMER [1985a, 1987a]).

We turn now to left fraction representations. A left fraction representation of a nonlinear system $\Sigma : S(\alpha^m) \to S(R^p)$ involves an integer $q > 0$, a subspace $S \subset S(R^q)$, and a pair of stable systems $G : \mathrm{Im}\,\Sigma \to S$ and $T : S(\alpha^m) \to S$, where G is invertible, such that $\Sigma = G^{-1}T$. The main use of left fraction representations in our context is for the purpose of parametrizing the set of pairs of stable systems A, B which satisfy an equation of the form $AP + BQ = M$. Here, P and Q originate from a right coprime fraction representation $\Sigma = PQ^{-1}$ of the same system Σ, and M is a fixed unimodular system. We have already indicated in (2.6) how such a parametrization may be obtained. The only questions that we still have to deal with in this context are the questions of the existence and of the construction of left fraction representations. The existence of left fraction representations for the systems we consider is guarantied by the following result, which we reproduce from HAMMER [1987a].

(3.13) THEOREM. An injective homogeneous system $\Sigma : S(\alpha^m) \to S(R^p)$ has a left fraction representation.

It is also quite easy to construct a left fraction representation for an injective homogeneous system $\Sigma : S(\alpha^m) \to S(R^p)$. Indeed, using the above mentioned fact that $\Sigma^{-1} : \mathrm{Im}\,\Sigma \to S(\alpha^m)$ is a stable system, and letting $I : S(\alpha^m) \to S(\alpha^m)$ be the identity map, the pair of stable systems

$$G := \Sigma^{-1} : \mathrm{Im}\,\Sigma \to S(\alpha^m),$$

(3.14)

$$T := I : S(\alpha^m) \to S(\alpha^m),$$

induces a left fraction representation $\Sigma = G^{-1}T$ (HAMMER [1987, section 4]).

To summarize, we see that a theory of fraction representations can be developed for nonlinear systems. This theory bears, in its external appearance, a close resemblance to the theory of fraction representations for transfer matrices of linear systems. The computations involved in the construction of fraction representations in the nonlinear case are relatively simple, and they become particularly simple for systems of the form Σ_γ, due to the simplicity of the inversion formula (3.4) for these systems. In our next section we discuss the problem of robust stabilization of nonlinear systems. The derivation of the results presented in the next section depends heavily on the theory of fraction representations that we have described here.

4. ROBUST STABILIZATION AND DYNAMICS ASSIGNMENT

In the present section we provide a survey of our results on the stabilization of nonlinear systems, following HAMMER [1987b and 1988]. In general, when considering stabilization of a system, one has to pay attention to three main issues - internal stability, dynamics assignment, and robustness.

Internal stability is a strong notion of stability, which is essential when the stability of composite systems is considered. In our case, internal stability of the configuration (3.2) means (i) that the configuration is input/output stable, (ii) that all the internal signals of the configuration are bounded, and (iii) that (i) and (ii) continue to hold when small noise signals are added to the signals at the points of entry of the subsystems Σ, π, φ, and γ of which the configuration consists. Only internally stable systems possess stable physical implementations. All composite systems that we construct below are internally stable.

The issue of dynamics assignment deals with the characterization of the dynamical properties that can be assigned to the internally stable closed loop (3.2), through proper choice of the compensators π, φ and γ. It provides the designer with the methodology to achieve a desired dynamical behviour for the final stabilized closed loop system. We show that, except for some obvious limitations, the stabilized closed loop system can be designed to have any desired dynamical behaviour, and we provide explicit constructions for compensators achieving that dynamical behaviour. In a qualitative way, the situation here is similar to the well known situation in the case of pole assignment for linear time invariant systems.

The issue of robustness deals with the stabilization of systems whose descriptions are not accurately known. Specifically, the situation in our case is as follows. Recall that the systems Σ whose stabilization we consider are given by recursive representations of the form

$$x_{k+1} = f(x_k, u_k),\ k = 0,1,2, \dots,$$

where the initial condition x_0 is specified. We shall consider the case where the recursion function f of the system that needs to be stabilized is not accurately known. Rather, a nominal recursion function f_n is given for the system Σ, and the actual recursion function f of the system may deviate from its nominal description. We assume that the actual recursion function is of the form

$$(4.1) \qquad f(x_k,u_k) = f_n(x_k,u_k) + \nu(x_k,u_k),$$

where the function ν describes the deviation from nominality. Of course, the function ν is not known, and, qualitatively speaking, we assume only that a bound on the magnitude of its parameters is given. We shall make the last statement more precise in the sequel. The fundamental question in the theory of robust stabilization can then be stated as follows. Assume that the nominal recursion function f_n of the system is given. Is it possible to design an internally stable control configuration that will stabilize the actual system Σ, irrespectively of the deviation function ν, as long as the latter is continuous and its parameters do not exceed a prespecified bound. If such a design is possible, how is it done.

We describe now our design procedure for the robust stabilization of nonlinear systems. The procedure allows dynamics assignment. At the end of the section we provide an explicit example on the computation of robustly stabilizing compensators, using our procedure. Throughout our review here we shall assume that the system $\Sigma : S(\alpha^m) \to S(R^p)$ that needs to be stabilized has an input space which is of the same dimension as its output space, namely, that $m = p$. This assumption simplifies the presentation, but is of no fundamental consequence in our framework. The general case where $m \neq p$ is treated in HAMMER [1987b, 1988].

Let then $\Sigma : S(\alpha^m) \to S(R^m)$ be the system that needs to be stabilized, and assume it is strictly causal and homogeneous. As we have discussed before, the basic system whose stabilization we shall consider is the system Σ_γ, which here takes the form

$$(4.2) \qquad \Sigma_\gamma = \gamma + \Sigma \ : S(\alpha^m) \to S(R^m),$$

where γ is an $m \times m$ nonsingular matrix. We recall that the system Σ_γ is bicausal. One of the basic steps in our stabilization procedure is to find an $m \times m$ nonsigular matrix γ for which the following condition is satisfied.

(4.3) CONDITION. <u>There is a real number</u> $\delta > 0$ <u>such that</u> $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ <u>for some real number</u> $\alpha > 0$.

In qualitative terms, the matrix γ shifts the image of the system so as to include a subspace of the form $S(\delta^m)$; the restriction of the inverse system $\Sigma_\gamma^{-1} : S(\delta^m) \to S(\alpha^m)$ becomes then a stable system. The justification of these statements and explicit methods for the computation of γ for some common classes of recursive systems are given in HAMMER [1987b and 1988]. We comment here that stronger results can be obtained when γ is allowed to be a nonlinear dynamic system. However, quite general results on robust stabilization and dynamics assignment for nonlinear systems can be obtained even when γ is restricted to be an $m \times m$ nonsingular matrix, as we assume throughout our discussion here.

In our present context, the system Σ is not accurately known, and we have to study the effects of the uncertainty in the description of Σ on Condition (4.3). For this purpose we need to describe more accurately the nature of the deviation functions ν in (4.1). We do so by defining a 'neighbourhood' $\mathcal{S}(*,\Delta)$ of radius Δ around the nominal system Σ_n, which consists of all systems whose deviation from Σ_n is permissible. We describe $\mathcal{S}(*,\Delta)$ in terms of quantities directly related to the recursion functions, distinguishing between two different classes of recursion functions, as follows.

The first class of recursion functions we consider is the class of recursion functions with bounded nonlinearities. First, some notation. Given an $m \times m$ matrix A with entries a_{ij}, we denote $||A|| := \max\{|a_{ij}|, i,j = 1, ..., m\}$. Let $\Sigma_n : S(R^m) \to S(R^m)$ be our nominal system, having a recursive representation $x_{k+1} = f_n(x_k, u_k)$ with $x_o = 0$. Assume the nominal recursion function is of the form $f_n(x_k, u_k) = Fx + Gu + \psi(x,u)$, where F and G are $m \times m$ matrices, and where the function $\psi : R^m \times R^m \to R^m$ is continuous and bounded, say $|\psi(x,u)| \le N$ for all $x, u \in R^m$. Now, for a real number $\Delta > 0$, we define a class $\mathcal{S}(F,G,\Delta)$ of systems that deviate 'by Δ' from the nominal system Σ_n. Specifically, $\mathcal{S}(F,G,\Delta)$ consists of all systems $\Sigma : S(R^m) \to S(R^m)$ having recursive representations $x_{k+1} = f_n(x_k, u_k) + \nu(x_k, u_k)$, $x_o = 0$, with the deviation function $\nu : R^m \times R^m \to R^m$ being of the form $\nu(x,u) = \Gamma x + \Lambda u + \psi_\nu(x,u)$, where Γ and Λ are $m \times m$ matrices satisfying $\|\Gamma\| \le \Delta$ and $\|\Lambda\| \le \Delta$, and where $\psi_\nu : R^m \times R^m \to R^m$ is a bounded continuous function, say $|\psi_\nu(x,u)| \le N$ for all $x, u \in R^m$. As usual, we say that a linear system is stabilizable if all its unreachable modes correspond to eigenvalues having absolute value strictly less than one. We then have the following result, which guaranties the existence of an $m \times m$ nonsingular matrix γ satisfying Condition (4.3) for all our deviated systems.

(4.4) PROPOSITION. Let $\mathcal{S}(F,G,\Delta)$ be the class of systems $\Sigma : S(R^m) \to S(R^m)$ defined in the previous paragraph, and assume that the pair F, G is stabilizable. Then, there is a real number $\Delta > 0$ and an $m \times m$ nonsingular matrix γ such that the following holds true. For every real number $\delta > 0$, there is a real number $\alpha > 0$ satisfying $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for all systems $\Sigma \in \mathcal{S}(F,G,\Delta)$.

The second class of systems we consider is more general than the one considered in Proposition (4.4), and it consists of systems having recursion functions which are differentiable. The results for this more general class of systems are somewhat weaker in the sense that the real number δ can no longer be chosen arbitrarily large. Nevertheless, robust stabilization with dynamics assignment can still be achieved for this rather general class of systems. Again, let $\Sigma_n : S(R^m) \to S(R^m)$ be our nominal system, having a recursive representation $x_{k+1} = f_n(x_k,u_k)$ with $x_0 = 0$. Assume that the nominal recursion function f_n is differentiable at the origin and that $f(0,0) = 0$, and let (F,G), where F and G are $m \times m$ matrices, be the Jacobian matrix of the partial derivatives of f_n at the origin. Now, given a real number $\Delta > 0$, we define a class $\mathcal{S}(f_n,\Delta)$ of systems that deviate 'by Δ' from the nominal system Σ_n. First, we fix a neighbourhood $\mathcal{N}$ of the origin and a real number $N > 0$. Then, $\mathcal{S}(f_n,\Delta)$ consists of all systems $\Sigma : S(R^m) \to S(R^m)$ having recursive representations of the form $x_{k+1} = f_n(x_k,u_k) + \nu(x_k,u_k)$, $x_0 = 0$, where the deviation function $\nu : R^m \times R^m \to R^m$ satisfies the following conditions. (i) ν is twice continuously differentiable over $\mathcal{N}$, and all its second order partial derivatives there are bounded in absolute value by N; (ii) $\nu(0,0) = 0$; and (iii) the Jacobian matrix (Γ,Λ) of the partial derivatives of ν at the origin, partitioned into the $m \times m$ matrices Γ and Λ, satisfies $\|\Gamma\| < \Delta$ and $\|\Lambda\| < \Delta$.

(4.5) PROPOSITION. Let $\mathcal{S}(f_n,\Delta)$ be the class of systems $\Sigma : S(R^m) \to S(R^m)$ defined in the previous paragraph. Let (F,G), where F and G are $m \times m$ matrices, be the

<u>Jacobian matrix of the partial derivatives of the nominal recursion function</u> f_n <u>at the origin, and assume that the pair</u> F, G <u>is stabilizable. Then, there are real numbers</u> $\Delta, \delta, \alpha > 0$ <u>and an</u> $m \times m$ <u>nonsingular matrix</u> γ <u>such that</u> $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ <u>for all systems</u> $\Sigma \in \mathcal{S}(f_n, \Delta)$.

As an example of a class of systems satisfying the conditions of Proposition(4.5), consider the following single-input single-output case. Let the nominal system Σ_n : $S(R) \to S(R)$ be given by the recursive representation $x_{k+1} = 2Exp(x_k + u_k) - 2 =: f_n(x_k, u_k)$. Now, fix some real number $N > 0$. Then, the class of systems $\Sigma : S(R) \to S(R)$ having recursive representations of the form $x_{k+1} = 2Exp(x_k + u_k) - 2 + ax_k + bx_k^2 + cu_k + du_k^2 + gx_ku_k$, where $|a|, |c| < \Delta$ and $|b|, |d|, |g| < N/2$, is a class of systems contained in $\mathcal{S}(f_n, \Delta)$, and hence the Proposition applies to it.

We remark that in HAMMER [1988] we described explicit ways for the computation of nonsingular matrices γ satisfying the conditions of Propositions (4.4) and (4.5). Once the matrix γ is at our disposition, we can directly proceed to the construction of the stabilizing compensators π and φ in configuration (3.2). We provide now a step by step description of the construction of compensators that robustly stabilize our system, and allow for assignment of dynamical properties for the final internally stable closed loop.

Let $\Sigma_n : S(R^m) \to S(R^m)$ be the given nominal system, and let $x_{k+1} = f_n(x_k, u_k)$ be its recursive representation, with the initial condition $x_0 = 0$. As before, we use the notation $\mathcal{S}(*, \Delta)$ for a 'neighbourhood' of 'radius' Δ of the system Σ, by which we simply mean a generic notation, referring to one of the sets $\mathcal{S}(F,G,\Delta)$ or $\mathcal{S}(f_n, \Delta)$ mentioned in Propositions (4.4) or (4.5). We shall assume that the given nominal recursion function f_n satisfies all the conditions involved in the use of these sets of systems, so that, when $\mathcal{S}(*, \Delta)$ is $\mathcal{S}(F,G,\Delta)$, the pair F, G is stabilizable; and when $\mathcal{S}(*, \Delta)$ is $\mathcal{S}(f_n, \Delta)$, the Jacobian matrix (F,G) of f_n at the origin, when partitioned into the pair of $m \times m$ matrices F and G, yields a stabilizable pair.

Our stabilization procedure consists then of the following steps.

Step 1. Choose a real number $\theta > 0$. This number will serve as the bound on the amplitude of the input sequences of the final stabilized closed-loop system. The choice of the number θ is usually determined by practical considerations, and there are no theoretical restrictions on its choice.

Step 2. Find a constant $m \times m$ nonsingular matrix γ for which there are three real numbers $\Delta, \delta, \alpha > 0$ such that the condition $S(\delta^P) \subset \Sigma_\gamma[S(\alpha^P)]$ holds for all systems $\Sigma \in \mathcal{S}(*,\Delta)$. The existence of such a matrix γ is guarantied by Propositions (4.4) and (4.5). Explicit methods for the computation of the matrix γ are described in HAMMER [1988].

Step 3. Choose a positive number ξ, and, using the numbers θ, δ, and α of the previous steps, choose constant positive numbers $\varsigma < \delta$ and $\varepsilon < \min\{\theta/\alpha, \xi/(2\alpha)\}$.

Step 4. Choose a recursive, unimodular, bicausal, and uniformly ℓ^∞-continuous system $M : S(R^m) \to S(R^m)$. The system M will determine the dynamical behavior of the closed loop system, as in (2.5) (see also HAMMER [1987b]). An elementary possible choice for M is $M := \beta I$, where $I : S(R^m) \to S(R^m)$ is the identity system and β is a nonzero constant.

Step 5. Find a real number $c > 0$ so that the system $M' := Mc$ satisfies the condition $M'^{-1}[S((5\theta + \xi)^m)] \subset S(\varsigma^m)$. This is simply a scaling operation which has no dynamical implications, and is performed as follows. In view of the fact that M is unimodular, the system M^{-1} is stable, and, consequently, there is a real number $\lambda > 0$ satisfying $M^{-1}[S((5\theta + \xi)^m)] \subset S(\lambda^m)$. But then, taking $c := (\lambda/\varsigma)$, we obtain that the system $M' := Mc$ satisfies $M^{-1}[S((5\theta + \xi)^m)] \subset S(\varsigma^m)$. If we use the choice $M = \beta I$ mentioned in Step 4, then, for $\beta \geq (5\theta+\xi)/\varsigma$, we obtain directly $M^{-1}[S((5\theta+\xi)^m)] \subset S(\varsigma^m)$.

Step 6. Construct the static system $E : S(R^m) \to S(\varsigma^m)$ given by the representation

(4.6) $\quad E: \ y_k := e(u_k), \quad k = 0, 1, 2, \ldots\ldots,$

$y = Eu,$

where e is a function $R^m \to [-\varsigma,\varsigma]^m$ defined as follows. For every vector $x = (x_1, \ldots, x_m) \in R^m$, it takes the value $e((x_1, \ldots, x_m)) := (\alpha_1, \ldots, \alpha_m)$, where $\alpha_i := x_i$ if $|x_i| \le \varsigma$ and $\alpha_i := \varsigma\,\mathrm{sign}(x_i)$ if $|x_i| > \varsigma$, and where $\mathrm{sign}(\cdot)$ is ± 1, depending on the sign of the argument. It is clear that the system E is recursive, causal, stable, and uniformly ℓ^∞-continuous, and it is in fact an extension of the identity system $I : S(\varsigma^m) \to S(\varsigma^m)$.

Step 7. Using the nominal system Σ_n, we construct the system $\Sigma_{n\gamma} := \gamma + \Sigma_n$ and its inverse $\Sigma_{n\gamma}^{-1}$, which exists by virtue of the bicausality of $\Sigma_{n\gamma}$. Using (3.4) and the fact that γ is invertible, we obtain an explicit recursive representation for $\Sigma_{n\gamma}^{-1}$, given by

$$u_{k+1} = \gamma^{-1}\{z_{k+1} - f_n((z-\gamma u)_k, u_k)\}\ ,\ k = 0,1,2, \ldots,$$

$$u_o = \gamma^{-1}\{z_o - x_o\},$$

where x_o is the given initial condition of the system Σ_n, z is the input sequence of $\Sigma_{n\gamma}^{-1}$, and u is the output sequence of $\Sigma_{n\gamma}^{-1}$. We shall only be interested in the restriction $\Sigma_{n\gamma}^{-1} : S(\delta^m) \to S(\alpha^m)$, which, as we mentioned earlier, is a stable and bicausal system.

Step 8. Combining the results of Steps 6 and 7, we construct the system

$$Q_{n*} := \Sigma_{n\gamma}^{-1}E \ : S(R^m) \to S(\alpha^m)$$

as a composition of the two recursive systems $\Sigma_{n\gamma}^{-1}$ and E. The system Q_{n*} is stable and causal, and it can be readily implemented on a digital computer.

Step 5. Construct the two systems

$$A := M' - \varepsilon Q_{n*} : S(R^m) \to S(R^m),$$

$$B := \varepsilon I \ : S(R^m) \to S(R^m),$$

where $I : S(R^m) \to S(R^m)$ is the identity system and ε is from step 3. From these, using (2.2), construct the compensators

$$\pi = (1/\varepsilon)I \quad : S(R^m) \to S(R^m),$$

(4.7)

$$\varphi = M' - \varepsilon Q_{n*} : S(R^m) \to S(R^m).$$

Notice that the precompensator π here is simply an amplifier with amplification factor of $1/\varepsilon$.

Steps 1 to 8 complete the construction of compensators π and φ which, when connected in the closed loop $\Sigma_{(\gamma,\pi,\varphi)}$, yield robust stabilization of the system Σ. The closed loop configuration $\Sigma_{(\gamma,\pi,\varphi)}$ will be internally stable for any system $\Sigma \in \mathcal{S}(*,\Delta)$. According to our selection in Step 1, the input sequences to $\Sigma_{(\gamma,\pi,\varphi)}$ must be taken from $S(\theta^m)$. In our construction of the compensators π and φ we have used only the given nominal recursion function f_n, and the compensators we derived are given in explicit form and are implementable. We can achieve desirable dynamics assignment for the stabilized closed loop system through the selection of the unimodular system M in Step 4. Of course, the exact input/output relationship induced by the closed loop configuration depends on the particular system Σ inserted in it. Detailed proofs and justification for the design procedure we have outlined here are given in HAMMER [1987b and 1988], where more general forms of stabilizing compensators are also described.

We conclude this note with a rather simple example on the computation of compensators π and φ that yield robust stabilization of a given nominal system. The example is reproduced from HAMMER [1988], where all the computations are described in detail. Here, we only exhibit the class of systems that is stabilized and the final form of the compensators, in order to provide a feeling of the explicit form of the solutions provided by procedure described before.

(4.8) EXAMPLE. We consider the design of a robust stabilization scheme for the following class of single-input single-output systems. The nominal system $\Sigma_n : S(R) \to S(R)$ is given by the recursive representation

$$\Sigma_n : \quad x_{k+1} = 2x_k + 2u_k + \sin(x_k u_k), \; x_o = 0.$$

The disturbed system Σ belongs to the class of systems having recursive representations of the form

$$\Sigma : \quad x_{k+1} = (2+\kappa)x_k + (2+\lambda)u_k + \sigma\sin(x_k u_k),\ x_0 = 0,$$

where κ, λ, and σ are real numbers in the intervals

$$-⅓ \le \kappa \le ⅓,\quad -⅓ \le \lambda \le ⅓,\ \text{and}\ \ -1 \le \sigma \le 1.$$

Our objective is to design an internally stable configuration that will stabilize any system of this class. We use only the parameters of the nominal recursion function. We note that γ is a scalar here. In HAMMER [1988] we have used the following values for the design parameters : $\theta = 2$, $\gamma = 1$, $\delta = 1$, $\alpha = 19$, $\xi = 2$, $\varsigma = 1/2$, and $\varepsilon = 1/20$. We take the simple choice $M = \beta I$ for our unimodular system, with $\beta = 25$, so that, recalling that we have a scalar system here, $M = 25$.

The system Q_{n*} here can be readily computed, and, denoting by $\{x_k\}$ the input sequence of Q_{n*} and by $\{z_k\}$ the output sequence of Q_{n*}, the representation of Q_{n*} is given by the relations

$$Q_{n*} : \begin{cases} y_k := e(x_k), \\ z_{k+1} = y_{k+1} - 2y_k - \sin[(y_k - z_k)z_k], \\ z_0 = y_0, \end{cases}$$

$k = 0,1,2, \ldots$. Here, the function $e : R \to [-1/2, 1/2]$ is defined by $e(x) = x$ if $|x| \le 1/2$ and $e(x) = (1/2)\text{sign}(x)$ if $|x| > 1/2$, where $\text{sign}(x) = \pm 1$, depending on the sign of x. The compensators become

$$\varphi = 25 - (1/20)Q_{n*},$$

$$\pi = 20.$$

Then, for any system $\Sigma : S(R) \to S(R)$ having a recursive representation of the form

$x_{k+1} = (2+\kappa)x_k + (2+\lambda)u_k + \sigma\sin(x_k u_k)$, where $-1/3 \le \kappa \le 1/3$, $-1/3 \le \lambda \le 1/3$, and $-1 \le \sigma \le 1$, the closed loop $\Sigma_{(1,\pi,\varphi)}$ around Σ will be internally stable for all input sequences from $S(2)$. As we see, the compensators π and φ that we obtained can be readily implemented.

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