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# Feedback representation of precompensators†

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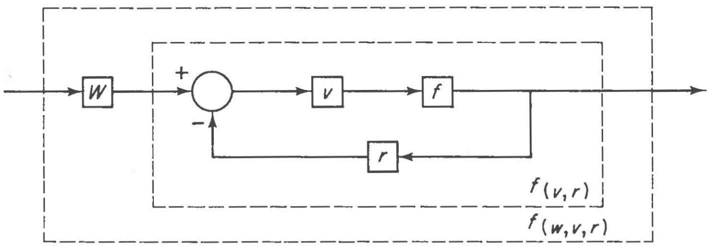
An algebraic theory is developed for the design of internally stable control configurations, which consist of dynamic output feedback, inside-loop precompensation, and outside-loop precompensation. The underlying quantity is an equivalent precompensator which relates the given system to the desired system, and which may induce unstable pole–zero cancellations. From this equivalent precompensator, the controllers of the final internally stable control configuration are constructed. The main step in this construction involves a partial-fraction decomposition of matrices. The problem of minimizing the outside-loop precompensator (without affecting the transfer matrix of the final system) is considered. For injective systems, the outside-loop precompensator can be eliminated in almost all cases.

## 1. Introduction

Let $f$ be the transfer matrix of a given linear time-invariant system, and assume that one is required to design a control configuration that will transform $f$ into a specified transfer matrix $f'$. The desired transfer matrix $f'$ is usually obtained through simulation, or through various optimization methods.




The control configuration we are interested in is that shown in the Figure, the classical configuration consisting of precompensators and feedback. (Due to practical limitations, postcompensators—i.e. dynamical transformations of the system outputs—are undesirable in many cases.) Here, $w$ is a causal (external) precompensator, $v$ is a causal (internal) precompensator, and $r$ is a causal output feedback compensator. The maps $f_{(v,r)}$ and $f_{(w,v,r)}$ denote the transfer matrices of the respective composite systems, and we have evidently that $f_{(w,v,r)} = f_{(v,r)}w$.

A major issue in our discussion are stability considerations. We have to guarantee that the composite system is *internally stable*, that is, that all its modes, including the unobservable and the unreachable ones, are stable.

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Further, in order to preserve the existing degrees of freedom in the control variables, we shall impose throughout our discussion the additional requirement that the precompensators $w$ and $v$ are non-singular. This will ensure that the resulting system $f_{(w,v,r)}$ has the same control capabilities as the original one $f$. We can now summarize the situation in the following classical manner (see also Newton *et al.* 1957).

*Feedback design problem*

Let $f$ and $f'$ be transfer matrices of linear time-invariant systems. Construct (if possible) causal compensators $w$, $v$ and $r$ such that the transfer matrices $f'=f_{(w,v,r)}$, and the composite system represented by $f_{(w,v,r)}$ is internally stable.

In order to simplify our statements, we shall use the notation

$$f' \stackrel{\sigma}{=} f_{(w,v,r)} \tag{1}$$

by which we mean that the transfer matrices $f'$ and $f_{(w,v,r)}$ are equal, and that the composite system represented by the right-hand side is internally stable.

An additional important class of design objectives related to the configuration in the Figure is motivated by sensitivity phenomena. Qualitatively, classical sensitivity considerations have shown that, under conditions of high forward gain, the parameters of $f_{(w,v,r)}$ are more sensitive to variations in the parameters of $w$ and $r$ than they are to variations in the parameters of $v$ and $f$. Thus, it is desirable to reduce the compensators $w$ and $r$, and to include as much as possible of the compensation dynamics in the (internal) precompensator $v$. This will lead to a design which contains fewer critical parameters.

In the present paper we construct compensators $w$, $v$ and $r$ in solving the feedback design problem. We pay particular attention to the minimization of the external precompensator $w$. In fact, we show that when $f$ is injective, $w$ can be eliminated in almost all cases, so that, generically, only the internal precompensator $v$ and the feedback $r$ are needed. The formalism developed here also allows us to obtain a design in which the dynamical orders of both $w$ and $r$ are minimal, and this topic is discussed in Hammer (1981 c, 1983 c).

Our approach to the feedback design problem can be qualitatively summarized as follows. Let $f$ be the transfer matrix of the given system, and let $f'$ be a stable desired transfer matrix. As a preliminary, we show that the following statements (*a*) and (*b*) are equivalent: (*a*) There exist causal compensators $w$, $v$ and $r$, where $w$ and $v$ are non-singular, such that $f' \stackrel{\sigma}{=} f_{(w,v,r)}$. (*b*) There exists a non-singular, causal, and stable precompensator $l$ such that $f'=fl$. Of course, in general, the combination $fl$ cannot be physically implemented as such, owing to the lack of internal stability. Now, given the transfer matrices $f$ and $f'$, conditions for the existence of a non-singular, causal and stable precompensator $l$ satisfying $f'=fl$ are known (Morse 1975, Hammer 1981 a, Hammer and Heymann 1983 b). Moreover, in case $l$ exists, its computation is relatively simple, and it can be accomplished through the employment of a suitable generalized inverse of $f$ (Hammer and Heymann 1983 b). Thus, the feedback design problem can be equivalently reduced to the following.

*Feedback representation problem*

Let $l$ be a non-singular, causal and stable precompensator for which $fl$ is stable. Find non-singular causal precompensators $w$ and $v$, and a causal feedback $r$, such that $fl \overset{\sigma}{=} f_{(w,v,r)}$. As an additional requirement, minimize the dynamical order of $w$.

Our solution to the feedback representation problem is as follows. First, we derive from the given transfer matrix $f$ a pair of non-singular matrices $D_\sigma$ and $D_0$, the first of which is determined by the unstable zeros and by the infinite zeros of $f$, and the second of which is determined by the unstable poles of $f$. Now, let $l$ be the desired precompensator. The main step in the solution (see § 5) is to compute a matrix partial-fraction decomposition

$$(D_\sigma l)^{-1} = PA^{-1} + QB^{-1} \tag{2}$$

which, in addition to usual minimality conditions, has to satisfy the following: the matrices $A$ and $B$ are left coprime ; $D_\sigma$ is a left divisor of $A$ ; and $D_0$ is a left divisor of $B$.

The partial-fraction decomposition (2) exists for *almost every* admissible precompensator $l$ (i.e. generically), and, in almost all cases (e.g. when unstable zeros of $l$ do not coincide with unstable zeros of $f$), its computation is straightforward. When the decomposition (2) exists (that is, generically) and the system $f$ is injective, the external precompensator $w$ is not required, and compensators $v$ and $r$ satisfying $fl \overset{\sigma}{=} f_{(v,r)}$ are directly determined by it. Roughly speaking, $v^{-1}$ is proportional to $PA^{-1}$ and $r$ is proportional to $QB^{-1}$

$$v^{-1} \sim PA^{-1}, \quad r \sim QB^{-1} \tag{3}$$

All the possible partial-fraction decompositions of the form (2) characterize all the pairs $\{v, r\}$ satisfying $fl \overset{\sigma}{=} f_{(v,r)}$ (through (3)) so that we also have a characterization of all such pairs. Being an additive condition, (2) (and (3)) give a direct insight into the relationship between $v$ and $r$ (for a fixed $l$), and into the effect on $v$ of different choices of $r$. In particular, the minimization of the feedback $r$ is obtained by minimizing the matrix $B$, subject to the above conditions (Hammer 1981 c, 1983 c).

Direct investigations into internal stability of multivariable linear control systems have received considerable attention in the system theoretic literature. In the last decade, regulation with internal stability was studied by Wonham and Pearson (1974), Wonham (1974), Davison (1976), and Cheng and Pearson (1978). Feedback interconnections were studied by Desoer and Chan (1975), Desoer and Vidyasagar (1975), and Rosenbrock and Hayton (1977). The so called ' output regulation problem ' was treated by Wolovich and Ferreira (1979). The design of optimal controllers was investigated by Youla *et al.* (1976). More recently, questions related to internal stability and feedback were considered by Desoer *et al.* (1980), Pernebo (1981), Desoer and Chen (1981), Zames (1981), Francis and Vidyasagar (1980), and Hammer (1981 b, 1983 a).

We shall outline the organization of the present paper at the end of § 2, after introducing some notation.

## 2. Preliminary considerations

Let $K$ be a field, and let $S$ be a $K$-linear space. We denote by $\Lambda S$ the set of all formal Laurent series with coefficients in $S$, of the form

$$s = \sum_{t=t_0}^{\infty} s_t z^{-t} \tag{4}$$

where for all $t$, $s_t \in S$. Then, under the operations of coefficient-wise addition and convolution as scalar multiplication, the set $\Lambda K$ is endowed with a field structure, and the set $\Lambda S$ forms a linear space over $\Lambda K$. Moreover, if the $K$-linear space $S$ is finite-dimensional, then so also is $\Lambda S$ as a $\Lambda K$-linear space, and $\dim_{\Lambda K} \Lambda S = \dim_K S$.

Further, let $U$ and $Y$ be finite-dimensional $K$-linear spaces, and let $\Sigma$ be a $K$-linear time-invariant system, admitting input values from $U$ and having its output values in $Y$. Assume also that $\Sigma$ possesses a transfer matrix $T$. Then, clearly, $T$ has its entries in the field $\Lambda K$, and, thus, induces in a natural way a $\Lambda K$-linear map $f_T : \Lambda U \to \Lambda Y$. Conversely, let $f : \Lambda U \to \Lambda Y$ be a $\Lambda K$-linear map. Then, $f$ can, of course, be represented as a matrix, relative to specified bases $u_1, \ldots, u_m$ in $\Lambda U$ and $y_1, \ldots, y_p$ in $\Lambda Y$. Of particular importance is the case when $u_1, \ldots, u_m$ belong to $U$ and $y_1, \ldots, y_p$ belong to $Y$, where $U$ and $Y$ are regarded as subsets of $\Lambda U$ and $\Lambda Y$, respectively. In this case the matrix representation $Z_f$ of $f$ is called a *transfer matrix*, and, if $f = f_T$, we clearly have that $Z_f$ coincides with $T$. Thus, a transfer matrix and a $\Lambda K$-linear map are equivalent quantities. (For a more abstract interpretation in the discrete-time case, see Kalman *et al.* (1969) and Wyman (1972).) Throughout our discussion, *all matrix representations will tacitly be assumed to be transfer matrices*. No sharp distinction between a map and its transfer matrix will be made.

Before continuing with our discussion of $\Lambda K$-linear maps, we need to review the underlying structure of the space $\Lambda S$. First, we denote by $\Omega^+ S$ the set of all (polynomial) elements in $\Lambda S$ of the form $s = \sum_{t=t_0}^{0} s_t z^{-t}$, $t_0 \leq 0$, and by $\Omega^- S$ the set of all (power series) elements in $\Lambda S$ of the form $s = \sum_{t=0}^{\infty} s_t z^{-t}$. Then, in particular, both of the sets $\Omega^+ K$ and $\Omega^- K$ are endowed with a principal ideal domain structure under the operations defined in $\Lambda K$. The set $\Omega^+ S$ forms a free $\Omega^+ K$-module, and the set $\Omega^- S$ forms a free $\Omega^- K$-module, both under the operations defined in $\Lambda S$. Moreover, when $S$ is finite-dimensional, then $\mathrm{rank}_{\Omega^+ K}\, \Omega^+ S = \dim_K S$, and $\mathrm{rank}_{\Omega^- K}\, \Omega^- S = \dim_K S$.

Next, let $f : \Lambda U \to \Lambda Y$ be a $\Lambda K$-linear map. Then, $f$ is called a *polynomial map* if all the entries in its transfer matrix are polynomials (i.e. in $\Omega^+ K$). Also, $f$ is a *causal map* if all the entries in its transfer matrix belong to $\Omega^- K$. Equivalently, $f$ is polynomial if and only if $f[\Omega^+ U] \subset \Omega^+ Y$, and $f$ is causal if and only if $f[\Omega^- U] \subset \Omega^- Y$ (Hammer and Heymann 1983 a). A $\Lambda K$-linear map $f : \Lambda U \to \Lambda Y$ is called *rational* if there exists a non-zero polynomial $\psi \in \Omega^+ K$ such that $(\psi f)$ is a polynomial map.

In discussion of causality it is sometimes convenient to employ the following classical notion of order. Let $s = \sum s_t z^{-t}$ be an element in $\Lambda S$. The *order* of $s$ is defined as $\mathrm{ord}\, s := \min_t \{s_t \neq 0\}$ if $s \neq 0$, and $\mathrm{ord}\, s := \infty$

if $s=0$. The *leading coefficient* $\hat{s}$ of $s$ is then defined as $\hat{s} := s_{\text{ord } s}$ if $s \neq 0$, and $\hat{s} := 0$ if $s=0$. In this terminology, a $\Lambda K$-linear map $f: \Lambda U \to \Lambda Y$ is causal if and only if $\text{ord } fu \geqslant \text{ord } u$ for all elements $u \in \Lambda U$. Several further related definitions are useful. A $\Lambda K$-linear map $f: \Lambda U \to \Lambda Y$ is *strictly causal* if $\text{ord } fu > \text{ord } u$ for all $u \in \Lambda U$. The map $f$ is called a *linear i/o map* if it is both strictly causal and rational. Finally, a $\Lambda K$-linear map $l$: $\Lambda U \to \Lambda U$ is *bicausal* if it is causal and if it possesses an inverse which is also causal (Hautus and Heymann 1978).

We turn next to proper bases. Let $s_1, \ldots, s_n \in \Lambda S$ be a set of elements. Then $s_1, \ldots, s_n$ are *properly independent* if their leading coefficients $\hat{s}_1, \ldots, \hat{s}_n (\in S)$ are $K$-linearly independent. A basis consisting of properly independent elements is called a *proper basis*. It can be shown that every $\Lambda K$-linear subspace $R \subset \Lambda U$ has a proper basis (Hammer and Heymann 1981). The importance of proper bases is related to the fact that they allow a finitary characterization of causality as follows. Let $u_1, \ldots, u_m$ be a proper basis of $\Lambda U$. Then, a $\Lambda K$-linear map $f: \Lambda U \to \Lambda Y$ is causal if and only if $\text{ord } fu_i \geqslant \text{ord } u_i$ for all $i = 1, \ldots, m$ (Wolovich 1974, Hammer and Heymann 1983 a). Also, a $\Lambda K$-linear map $l: \Lambda U \to \Lambda U$ is bicausal if and only if $lu_1, \ldots, lu_m$ are properly independent, and $\text{ord } lu_i = \text{ord } u_i$ for all $i = 1, \ldots, m$. A proper basis $u_1, \ldots, u_m$ is *ordered* if $\text{ord } u_{i+1} \geqslant \text{ord } u_i$ for all $i = 1, \ldots, m-1$.

We next turn to some basic notions related to stability. As was noted by Morse (1975), questions related to stability are most naturally analysed in a ring-theoretic framework. We next review some terminology in this context from Hammer (1981 a).

Let $\sigma$ be a multiplicative set (of polynomials) in $\Omega^+ K$ (i.e. for every pair of elements $k_1, k_2 \in \sigma$, also $k_1 k_2 \in \sigma$). We say that $\sigma$ is a *stability set* if it satisfies (i) $0 \notin \sigma$, (ii) $\sigma$ contains a polynomial of degree one, that is, there is an element $\alpha \in K$ such that $(z + \alpha) \in \sigma$ (Morse 1975). We now choose a stability set $\sigma \subset \Omega^+ K$, and leave it fixed throughout our discussion. Given a $\Lambda K$-linear map $f$: $\Lambda U \to \Lambda Y$, we say that $f$ is *i/o (input/output) stable* (in the sense of $\sigma$) if there exists an element $\psi \in \sigma$ such that $\psi f$ is a polynomial map.

Further, we denote by $\Omega_\sigma K$ the set of all i/o stable elements in $\Lambda K$. Explicitly, $\Omega_\sigma K$ is the set of all elements $\alpha \in \Lambda K$ which can be expressed as a polynomial fraction $\alpha = \beta / \gamma$, with $\gamma$ belonging to the stability set $\sigma$. It can then be shown that, under the operations defined in $\Lambda K$, $\Omega_\sigma K$ is endowed with a *principal ideal domain* structure. Also, a $\Lambda K$-linear map $f: \Lambda U \to \Lambda Y$ is i/o stable if and only if all entries in its transfer matrix belong to $\Omega_\sigma K$. This fact will allow us to employ the highly developed theory of matrices with entries in a principal ideal domain when studying i/o stable maps.

Next, one can incorporate the restriction of causality into the stability framework through the use of an additional class of rings (introduced by Morse (1975)), as follows. Let $\Omega_\sigma^- K := \Omega_\sigma K \cap \Omega^- K$, that is, the set of all elements in $\Lambda K$ which are both i/o stable and causal. Then, it was shown by Morse (1975) that $\Omega_\sigma^- K$ forms a principal ideal domain under the operations defined in $\Lambda K$. Clearly, a $\Lambda K$-linear map $f: \Lambda U \to \Lambda Y$ is both causal and i/o stable if and only if all entries in its transfer matrix belong to $\Omega_\sigma^- K$, and we are again faced with a situation of matrices with entries in a principal ideal domain.

Several types of unimodular maps appear in our discussion, and we now review the terminology. We say that a $\Lambda K$-linear map $l: \Lambda S \to \Lambda S$ is $\Omega^+K$- (respectively $\Omega^-K$-, $\Omega_\sigma K$-, $\Omega_\sigma^- K$-) *unimodular* if $l$ has an inverse $l^{-1}$ and if both of $l$ and $l^{-1}$ are polynomial (respectively causal, i/o stable, both causal and i/o stable). In particular, an $\Omega^+K$-unimodular map is the usual polynomial unimodular map, and an $\Omega^-K$-unimodular map is the bicausal map.

We turn next to certain canonical representations of systems in the stability sense, following Hammer (1981 a). Let $N: \Lambda U \to \Lambda Y$ and $D: \Lambda U \to \Lambda Y'$ be i/o stable $\Lambda K$-linear maps. We say that $N$ and $D$ are *right $\sigma^+$-coprime* if there exist i/o stable $\Lambda K$-linear maps $A: \Lambda Y \to \Lambda U$ and $B: \Lambda Y' \to \Lambda U$ such that $AN+BD=I$ (the identity map). We say that an i/o stable map $C: \Lambda U \to \Lambda U$ is a *common right $\sigma^+$-divisor* of $N$ and $D$ if there exist i/o stable maps $N': \Lambda U \to \Lambda Y$ and $D': \Lambda U \to \Lambda Y'$ such that $N=N'C$ and $D=D'C$. Now, let $f: \Lambda U \to \Lambda Y$ be a rational $\Lambda K$-linear map. A (matrix fraction) representation of the form $f=ND^{-1}$, where $N: \Lambda U \to \Lambda Y$ and $D: \Lambda U \to \Lambda U$ are i/o stable, is called a *right stability representation* of $f$. In case $N$ and $D$ are right $\sigma^+$-coprime, we say that this stability representation is *canonical*. Left stability representations are defined in a dual way. It can be shown that every rational $\Lambda K$-linear map has both right and left canonical stability representations.

If $f=ND^{-1}$ is a right canonical stability representation, then we say that $D$ is a *right $\sigma^+$-denominator* of $f$. It is worthwhile to note that $f$ is i/o stable if and only if its right $\sigma^+$-denominators are $\Omega_\sigma K$-unimodular.

Two particular types of canonical stability representations are distinguished by their minimality properties. One of these representations characterizes the unstable poles of the system, and the other one—the unstable zeros. Let $f: \Lambda U \to \Lambda Y$ be a rational $\Lambda K$-linear map. A right stability representation $NP^{-1}$ of $f$ is called a *right pole representation* whenever the following hold: (i) $P$ is a polynomial map, and (ii) if $f=RQ^{-1}$ is any right stability representation with $Q$ polynomial, then $P$ is a polynomial left divisor of $Q$. The matrix $P$ is then called a *right pole matrix* of $f$, and it characterizes the unstable poles of $f$ (Hammer 1981 a). Further, a right stability representation $f=ZD^{-1}$ is called a *right zero representation* whenever (i) $Z$ is a polynomial map, and (ii) if $RQ^{-1}$ is any right stability representation with $R$ polynomial, then $Z$ is a polynomial left divisor of $R$. The matrix $Z$ is then called a *right zeros matrix* of $f$, and it characterizes the unstable zeros of $f$ (see Hammer (1981 a) and also compare with Pernebo (1981)). Left pole and zero representations are defined in a dual way. It can be shown that pole and zero representations exist, and are canonical stability representations (Hammer 1981 a).

A right pole representation is constructed as follows. Let $f=RT^{-1}$ be a right coprime polynomial matrix fraction representation. One factors $T=PT_1$ into a multiple of polynomial matrices, where $T_1^{-1}$ is i/o stable, and where det $P$ is polynomially coprime with every element in the stability set $\sigma$. Then, letting $N := RT_1^{-1}$, it can be shown that $f=NP^{-1}$ is a right pole representation of $f$. A right zero representation is constructed dually—one factors $R=ZR_1$ into a multiple of polynomial matrices, where $R_1$ is square non-singular and $R_1^{-1}$ is i/o stable, and where the invariant factors of $Z$ are polynomially coprime with every element in $\sigma$. Then, denoting $D := TR_1^{-1}$, it can be shown that $f=ZD^{-1}$ is a right zero representation of $f$.

When considering pole and zero representations, it is convenient to employ the following type of matrices. Let $P : \Lambda U \to \Lambda U$ be a polynomial matrix. We say that $P$ is *completely unstable* (in the sense of $\sigma$) if the invariant factors of $P$ are (polynomially) coprime with every element in $\sigma$. It can then be seen that a canonical stability representation $f = NP^{-1}$ is a pole representation if and only if $P$ is a completely unstable polynomial map. The situation for zero representations is, of course, analogous. The following is a useful technical property of completely unstable polynomial maps, which can be easily verified (Hammer and Khargonekar 1981).

*Lemma* 2.1

*Let $R : \Lambda U \to \Lambda Y$ and $Q : \Lambda U \to \Lambda U$ be polynomial maps, and assume that $Q$ is non-singular and completely unstable. If the map $Q^{-1}R$ is i/o stable, then it is a polynomial map.*

Next, let $f = NP^{-1}$ be a right pole representation. Then, clearly, a right stability representation $f = N'P'^{-1}$ is a pole representation if and only if $P' = PM$, where $M : \Lambda U \to \Lambda U$ is polynomial unimodular. Thus, in particular, $\det P' = k \det P$, where $k \in K$. We now define the *pole degree* $\rho(f)$ of $f$ as

$$\rho(f) := -\operatorname{ord}(\det P)$$

When $K$ is the field of real numbers, then $\rho(f)$ is simply the number of unstable poles of $f$. Now, let $f = P_L^{-1} N_L$ be a left pole representation. We claim that

$$-\operatorname{ord}(\det P_L) = \rho(f) \tag{5}$$

Indeed, (5) is a direct consequence of the following.

*Theorem* 2.1

*Let $f : \Lambda U \to \Lambda Y$ be a rational $\Lambda K$-linear map, and let $NP^{-1}$ and $P_L^{-1} N_L$ be right and left canonical pole representations of $f$, respectively. Then, the polynomial matrices $P$ and $P_L$ have the same (non-trivial) invariant factors.*

*Proof of Theorem* 2.1

Let $M_1 : \Lambda Y \to \Lambda Y$ and $M_2 : \Lambda U \to \Lambda U$ be polynomial unimodular matrices such that $f_0 := M_1 f M_2$ is in the Smith–MacMillan canonical form (see, e.g. Rosenbrock (1970)). Then, since all non-zero entries in $f_0$ are on its main diagonal, $f_0$ clearly possesses a right and a left pole representation $N_0 P_0^{-1}$ and $P'_0{}^{-1} N'_0$, respectively, where $P_0$ and $P'_0$ are both diagonal matrices. But it is also clear that $P_0$ and $P'_0$ have the same non-trivial invariant factors, and, since $M_1$ and $M_2$ are polynomial unimodular, the latter implies our assertion. □

The following properties of the pole degree $\rho(\cdot)$ can be readily verified.

*Proposition* 2.1

*Let $f_1, f_2 : \Lambda U \to \Lambda Y$ and $f_3 : \Lambda Y \to \Lambda W$ be rational $\Lambda K$-linear maps. Then, the pole degrees satisfy the following:*

$$\text{(i)}\ \rho(f_1 + f_2) \leqslant \rho(f_1) + \rho(f_2) \quad \textit{and} \quad \text{(ii)}\ \rho(f_3 f_1) \leqslant \rho(f_3) + \rho(f_1)$$

In complete analogy, we also define the *zero degree* $\zeta(f)$ of a non-zero rational $\Lambda K$-linear map $f$, as follows. Let $f=ZD^{-1}$ be a canonical zero representation, and let $\psi_1, \ldots, \psi_n$ be the invariant factors of $Z$. Then, the zero degree is $\zeta(f):= -\operatorname{ord} \prod_{i=1}^{n} \psi_i$ (i.e. 'the number of unstable zeros'). The connection between zero degrees and pole degrees is given by the following proposition, which can be readily verified.

*Proposition* 2.2

*Let $f: \Lambda U \rightarrow \Lambda Y$ be a rational $\Lambda K$-linear map, and let $f=ND^{-1}$ be a right stability representation of $f$. Then, $\zeta(D) \geqslant \rho(f)$, and the stability representation $f=ND^{-1}$ is canonical if and only if $\zeta(D)=\rho(f)$.*

The pole degree, the zero degree and Proposition 2.2 are extensively employed in all our derivations in this paper.

A major role in our present discussion is played by certain notions related to the inversion of $\Lambda K$-linear maps. Let $f: \Lambda U \rightarrow \Lambda Y$ be a $\Lambda K$-linear map. We say that $f$ is (*left*) $\sigma^-$-*invertible* if there exists a causal and i/o stable $\Lambda K$-linear map $h: \Lambda Y \rightarrow \Lambda U$ such that $hf=I$, the identity. Evidently, every $\sigma^-$-invertible map is necessarily injective. Conversely, every injective (rational) $\Lambda K$-linear map can be made $\sigma^-$-invertible through composition with a suitable matrix, as follows. We first define a notion which is repeatedly used in our discussion. (Given i/o stable matrices $A, B: \Lambda U \rightarrow \Lambda U$, we say that $B$ is a *right* $\sigma^-$-*divisor* of $A$ if there exists a causal and i/o stable matrix $A'$ such that $A=A'B$.)

*Definition* 2.1

*Let $f: \Lambda U \rightarrow \Lambda Y$ be an injective rational $\Lambda K$-linear map, and let $D_\sigma: \Lambda U \rightarrow \Lambda U$ be a non-singular and i/o stable matrix. We say that $D_\sigma$ is a* ***σ-annihilator*** *of $f$ if it satisfies the following conditions:* (i) *$fD_\sigma^{-1}$ is left $\sigma^-$-invertible and* (ii) *for every non-singular and i/o stable matrix $D: \Lambda U \rightarrow \Lambda U$ for which $fD^{-1}$ is left $\sigma^-$-invertible, $D_\sigma$ is a right $\sigma^-$-divisor of $D$.*

Intuitively speaking, a $\sigma$-annihilator of $f$ is a 'minimal' i/o stable matrix $D_\sigma$ for which $fD_\sigma^{-1}$ is $\sigma^-$-invertible. It exactly cancels (or 'annihilates') the unstable zeros of $f$ and its zeros at infinity. Clearly, if $D_\sigma$ and $D'_\sigma$ are both $\sigma$-annihilators of $f$, then there exists an $\Omega_\sigma{}^-K$-unimodular map $l: \Lambda U \rightarrow \Lambda U$ such that $D'_\sigma = lD_\sigma$. Since $\sigma$-annihilators will be repeatedly employed throughout our discussion, we now describe in detail their construction.

*Construction of σ-annihilators*

Let $f: \Lambda U \rightarrow \Lambda Y$ be an injective rational $\Lambda K$-linear map, and let $f=D^{-1}N$ be a left coprime polynomial matrix fraction representation. We factor $N=N_sN_0$ into a multiple of polynomial matrices, where $N_0: \Lambda U \rightarrow \Lambda U$ is completely unstable and square non-singular, and where $N_s: \Lambda U \rightarrow \Lambda Y$ has no unstable zeros (i.e. all its invariant factors are divisors of elements in the stability set $\sigma$). Such a factorization can be obtained through a suitable factorization of the invariant factors in the Smith canonical form of $N$. Now, denote $g := D^{-1}N_s$, and let $M : \Lambda U \to \Lambda U$ be a polynomial unimodular matrix such that $g_s := gM$ has ordered and properly independent columns. Let $g_1, \ldots, g_m$ be the columns of $g_s$. The integers $\nu_i := \operatorname{ord} g_i$, $i = 1, \ldots, m$, are called the *σ-latency indices* of $f$. Define $g_0 := M^{-1}N_0 : \Lambda U \to \Lambda U$, so that $f = g_s g_0$, and note that $g_0$ is square non-singular. Finally, let $(z+\alpha)$ be a first-degree polynomial in $\sigma$, and define the square non-singular matrix

$$D_\sigma := [\operatorname{diag}((z+\alpha)^{-\nu_1}, \ldots, (z+\alpha)^{-\nu_m})]g_0 : \Lambda U \to \Lambda U \tag{6}$$

We next show that $D_\sigma$ is a $\sigma$-annihilator of $f$.

*Proposition* 2.3

*Let $f : \Lambda U \to \Lambda Y$ be an injective rational $\Lambda K$-linear map, and construct $D_\sigma$ to be as in* (6). *Then, $D_\sigma$ is a $\sigma$-annihilator of $f$.*

The proof of Proposition 2.3 depends on the following auxillary result, which is a consequence of the Hermite normal form theorem (see, for example, MacDuffee (1934)).

*Lemma* 2.2

*Let $f : \Lambda U \to \Lambda Y$ be a rational $\Lambda K$-linear map. Then there exists an $\Omega_\sigma^-K$-unimodular map $l : \Lambda Y \to \Lambda Y$ such that $lf = \begin{pmatrix} f_0 \\ 0 \end{pmatrix}$, where $f_0$ is surjective.*

*Proof of Proposition* 2.3

In view of Lemma 2.2 and the injectivity of $f$, there exists an $\Omega_\sigma^-K$-unimodular map $l : \Lambda Y \to \Lambda Y$ such that $lf = \begin{pmatrix} f_0 \\ 0 \end{pmatrix}$, where $f_0 : \Lambda U \to \Lambda U$ is square non-singular. By the construction of $D_\sigma$ in (6) it follows then that the matrix $l_0 := (f_0 D_\sigma^{-1})^{-1}$ is both bicausal and i/o stable. Then $f_0^{-1} = D_\sigma^{-1} l_0$, and, since by construction $\rho(f_0^{-1}) = \zeta(f_0) = \zeta(f) = \zeta(D_\sigma)$, it follows by the dual of Proposition 2.2 that $D_\sigma$ and $l_0$ are left $\sigma^+$-coprime.

Defining now $h := (l_0, 0)l : \Lambda Y \to \Lambda U$, we clearly have that $h$ is causal and i/o stable, and that $hfD_\sigma^{-1} = I$. Thus, $fD_\sigma^{-1}$ is $\sigma^-$-invertible, and part (i) of Definition 2.1 holds for $D_\sigma$. We consider next part (ii) of Definition 2.1. Let $D : \Lambda U \to \Lambda U$ be any non-singular and i/o stable matrix for which $fD^{-1}$ is $\sigma^-$-invertible. Then the matrix $\alpha := (f_0 D^{-1})^{-1}$ is both causal and i/o stable, and $\alpha = (l_0^{-1} D_\sigma D^{-1})^{-1} = DD_\sigma^{-1} l_0$. Whence, since $l_0$ is bicausal, $DD_\sigma^{-1}$ is causal. Also, since $D$ and $\alpha$ are both i/o stable, and since $D_\sigma$ and $l_0$ are left $\sigma^+$-coprime, it follows that $DD_\sigma^{-1}$ is also i/o stable. Thus, $DD_\sigma^{-1}$ is both causal and i/o stable, so that $D_\sigma$ is a right $\sigma^-$-divisor of $D$, and (ii) of Definition 2.1 hold for $D_\sigma$. This completes our proof. □

Using the expression $f_0 = l_0^{-1} D_\sigma$ derived in the above proof, and recalling that $l_0$ is bicausal, we directly obtain the following corollary.

*Corollary* 2.1

*Let $f : \Lambda U \to \Lambda Y$ be an injective $\Lambda K$-linear map. Then $f$ is strictly causal if and only if its $\sigma$-annihilator $D_\sigma$ is strictly causal.*

We conclude this section with a brief outline of the present paper. Let $f: \Lambda U \to \Lambda Y$ be a linear i/o map, and refer to the Figure. We have that

$$f_{(v,r)} = f l_{(v,r)}$$

where

$$l_{(v,r)} := v[I + rfv]^{-1}$$

is evidently a non-singular and causal equivalent precompensator, and

$$l_{(v,r)}{}^{-1} = v^{-1} + rf \tag{7}$$

From (7) it follows that the feedback representation problem involves (i) decomposition of a system into a sum (to obtain $v^{-1}$ and $(rf)$ from $l_{(v,r)}{}^{-1}$), and (ii) factorization into a series composition (to obtain $r$ from $(rf)$). In both cases, each factor or summand is required to satisfy certain causality and stability restrictions.

We study sum decomposition of systems in § 3, and § 4 is devoted to internal stability. The feedback representation problem is studied in § 5.

## 3. Decomposition into a sum

One of the more commonly used sum decompositions of transfer matrices is the polynomial truncation, which is formally defined as follows. Let $f: \Lambda U \to \Lambda Y$ be a transfer matrix, and let $f = \sum_{t=t_0}^{\infty} A_t z^{-t}$, where $A_t: U \to Y$ for all $t$, be an expansion of it into a formal Laurent series. Then, the polynomial truncation operator $\mathbf{L}^+$ is defined as

$$\mathbf{L}^+(f) := \sum_{t<0} A_t z^{-t} \quad \text{(the strictly polynomial part)} \tag{8}$$

so that $f - \mathbf{L}^+(f)$ is causal. Below, we examine a few additional types of sum decompositions of transfer matrices.

First, we consider an implication of causality on sum decompositions. Let $g: \Lambda U \to \Lambda U$ be a non-singular and causal $\Lambda K$-linear map, and consider any sum decomposition $g^{-1} = g_1 + g_2$, in which one of the summands, say $g_2$, is strictly causal. Then the following holds for $g_1$.

*Proposition* 3.1

*Let $g: \Lambda U \to \Lambda U$ be a non-singular and causal $\Lambda K$-linear map, and let $g^{-1} = g_1 + g_2$, where $g_2: \Lambda U \to \Lambda U$ is strictly causal. Then $g_1$ is non-singular, and $g_1{}^{-1}$ is causal.*

*Proof of Proposition* 3.1

Clearly, $g_1 = g^{-1}(I - gg_2)$, where, since $gg_2$ is strictly causal, the map $l := I - gg_2$ is bicausal. But then, $g_1 = g^{-1}l$ is non-singular, and $g_1{}^{-1} = l^{-1}g$ is causal. □

Sum decompositions of transfer matrices also appear in the stability context. We shall need to 'distribute' the unstable modes of a given transfer matrix among two summands in a prescribed way. This we do through partial-fraction decomposition of matrices. The classical theory of partial-fraction decompositions of scalar rational functions can be generalized to the matrix case in different ways, depending on the application at hand. One possibility is the following. Consider, for a moment, a polynomial matrix fraction $E := RQ^{-1}$. Applying scalar partial-fraction decomposition to the entries in the Smith–MacMillan canonical form of $E$, one obtains a decomposition $E = \sum R_i Q_i^{-1}$, where the characteristic polynomials of the matrices $Q_i$ are elementary divisors of $Q$ (Fuhrmann 1981). In our present discussion we use an alternative approach, and the term partial-fraction decomposition will have a somewhat different, though related, meaning.

Let $A, B : \Lambda U \to \Lambda U$ be non-singular and i/o stable maps, and consider the quantity $(AB)^{-1}$. Suppose that there exists a representation of the form

$$(AB)^{-1} = PC^{-1} + QD^{-1} \tag{9}$$

where all of $P, C, Q, D : \Lambda U \to \Lambda U$ are i/o stable. We refer to such a representation as a *partial-fraction decomposition*, and we say that it is *reduced* whenever the pairs $P, C$ as well as $Q, D$ are right $\sigma^+$-coprime.

Now, the matrices $C$ and $D$ have all their entries in the principal ideal domain $\Omega_\sigma K$, and, consequently (MacDuffee 1934), they possess a least-common right multiple $R$ over this ring. We shall call $R$ a *$\sigma^+$-LCRM* of $C$ and $D$. Explicitly, letting $D'C'^{-1}$ be a right canonical stability representation of $C^{-1}D$, we have $R = CD' = DC'$. Returning now to (9), we directly obtain

$$(AB)^{-1} = (PD' + QC')R^{-1} \tag{10}$$

and it follows that $(AB)$ is a left $\sigma^+$-divisor of $R$. Much of our interest in this problem is directed towards the particular case when $C$ and $D$ are left $\sigma^+$-coprime, and $AB = R$. This case can be characterized as follows.

*Proposition* 3.2

*Let $A, B : \Lambda U \to \Lambda U$ be non-singular i/o stable $\Lambda K$-linear maps, and let $(AB)^{-1} = PC^{-1} + QD^{-1}$ be a reduced partial-fraction decomposition. Then the following are equivalent:*

($\alpha$) *The pair $C$, $D$ is left $\sigma^+$-coprime, and has $AB$ as a $\sigma^+$-LCRM.*

($\beta$) *The zero degrees $\zeta(AB) = \zeta(C) + \zeta(D)$.*

*Proof of Proposition* 3.2

We use the notation of (10). First we note that, since $C'$ and $D'$ are right $\sigma^+$-coprime, $\zeta(C) \geqslant \zeta(C')$. Also, (*a*) $\zeta(C) = \zeta(C')$ if and only if $C, D$ are left $\sigma^+$-coprime. Further, we clearly have $\zeta(R) = \zeta(D) + \zeta(C')$. From (10), $\zeta(AB) \leqslant \zeta(R)$, and, since $AB$ is i/o stable, we have that $\zeta(AB) = \zeta(R)$ if and only if $\zeta(PD' + QC') = 0$. But, the latter holds if and only if $(PD' + QC')$ is $\Omega_\sigma K$-unimodular. Whence, it follows that (*b*) $\zeta(AB) = \zeta(R)$ if and only if $AB$ is a $\sigma^+$-LCRM of $C$ and $D$.

Now, if $\zeta(AB) = \zeta(C) + \zeta(D)$, then we have $\zeta(C) + \zeta(D) \leqslant \zeta(R) = \zeta(D) + \zeta(C') \leqslant \zeta(D) + \zeta(C)$, so that $\zeta(C) = \zeta(C')$ as well as $\zeta(AB) = \zeta(R)$. Thus, by (*a*) and (*b*) we obtain that ($\beta$) implies ($\alpha$). Conversely, if ($\alpha$) holds, then by (*a*) and (*b*), $\zeta(C) = \zeta(C')$ and $\zeta(AB) = \zeta(R)$, so that $\zeta(AB) = \zeta(R) = \zeta(D) + \zeta(C') = \zeta(D) + \zeta(C)$, and ($\beta$) holds as well. $\square$

In case $AB$ is a $\sigma^+$-LCRM of $C$ and $D$, then $D$ is evidently a left divisor of $AB$. Thus, in such a case, we can assume without loss of generality that $D=A$.

*Proposition* 3.3

*Let $A, B, C : \Lambda U \to \Lambda U$ be non-singular i/o stable $\Lambda K$-linear maps.*

(i) *If there exists a reduced partial-fraction decomposition $(AB)^{-1}=PC^{-1}+QA^{-1}$, then $AB$ is a $\sigma^+$-LCRM of $A$ and $C$.*

(ii) *Conversely, if $AB$ is a $\sigma^+$-LCRM of $A$ and $C$, then there exist i/o stable maps $P, Q : \Lambda U \to \Lambda U$ such that $(AB)^{-1}=PC^{-1}+QA^{-1}$.*

(iii) *In* (ii), *if $A$ and $C$ are left $\sigma^+$-coprime, then the pairs $C$, $P$ as well as $A$, $Q$ are right $\sigma^+$-coprime.*

*Proof of Proposition* 3.3

(i) Multiplying by $AB$, we obtain $PC^{-1}AB+QB=I$, so that $PC^{-1}AB$ is i/o stable. By the coprimeness of $P$ and $C$, it follows then that the map $G := C^{-1}AB$ is i/o stable as well. Thus, since $PG+QB=I$ and $CG=AB$, we obtain that $AB$ is a $\sigma^+$-LCRM of $A$ and $C$, and (i) follows. The proof of (ii) is by reversing the steps in the proof of (i).

In order to prove (iii), let $P'C'^{-1}$ and $Q'A'^{-1}$ be right canonical stability representations of $PC^{-1}$ and $QA^{-1}$, respectively. We note that (*) the zero degrees $\zeta(A') \leqslant \zeta(A)$ and $\zeta(C') \leqslant \zeta(C)$. Now, by the proof of Proposition 3.2 we have $\zeta(AB) \leqslant \zeta(A')+\zeta(C')$. Also, since $AB$ is a $\sigma^+$-LCRM of $A$ and $C$, there exists an i/o stable matrix $G$, right $\sigma^+$-coprime with $B$, such that $CG=AB$. But then, $C^{-1}A=GB^{-1}$ are canonical stability representations, and whence by (5) and Proposition 2.2, $\zeta(C)=\zeta(B)$. Thus, $\zeta(AB)=\zeta(A)+\zeta(B)=\zeta(A)+\zeta(C)$, so that, by a previous inequality, $\zeta(A)+\zeta(C) \leqslant \zeta(A')+\zeta(C')$. But then, (*) implies that $\zeta(A')=\zeta(A)$ and $\zeta(C')=\zeta(C)$, so that, by Proposition 2.2, the pairs $P$, $C$ as well as $Q$, $A$ are right $\sigma^+$-coprime. □

## 4. Internal stability

In this section we discuss the internal stability of the configuration shown in the Figure. When given the transfer matrices $w$, $v$ and $r$, explicit conditions for the internal stability of this configuration were derived by Desoer and Chan (1975). In the present paper we are interested in the converse problem, namely, in the construction of compensators $w$, $v$ and $r$ to meet specified requirements for the input–output behaviour of the final composite system. This construction will require a more detailed examination of internal stability, which we now proceed to do.

We start with a discussion of stability properties of series connections of systems. Let $g : \Lambda Y \to \Lambda Y'$ and $h : \Lambda U \to \Lambda Y$ be rational $\Lambda K$-linear maps, representing linear time-invariant systems $\Sigma_g$ and $\Sigma_h$, respectively. We assume that both of $\Sigma_g$ and $\Sigma_h$ have no hidden (i.e. unreachable or unobservable) modes. Yet, when $\Sigma_g$ and $\Sigma_h$ are combined in series, the resulting system $\Sigma_g\Sigma_h$ might, of course, possess hidden modes. We shall say that the composition $gh$ is *σ-detectable* if all hidden modes in $\Sigma_g\Sigma_h$ are stable modes. Equivalently, $gh$ is σ-detectable if and only if there occur no cancellations of unstable poles and unstable zeros, when the transfer matrices of $g$ and $h$ are multiplied. The latter is readily seen to be equivalent to the following proposition.

*Proposition* 4.1

*Let $g: \Lambda Y \to \Lambda Y'$ and $h: \Lambda U \to \Lambda Y$ be rational $\Lambda K$-linear maps, and let $g = D_L^{-1} N_L$ and $h = N_R D_R^{-1}$ be, respectively, left and right canonical stability representations. Then, $gh$ is $\sigma$-detectable if and only if* (i) *$D_L$ and $N_L N_R$ are left $\sigma^+$-coprime, and* (ii) *$N_L N_R$ and $D_R$ are right $\sigma^+$-coprime.*

It is intuitively clear that $gh$ is $\sigma$-detectable if and only if the number of unstable (canonical) poles of $gh$ is equal to the sum of the numbers of such poles in $g$ and in $h$. This fact, which can be readily verified, is stated in the following proposition.

*Proposition* 4.2

*Let $g: \Lambda Y \to \Lambda Y'$ and $h: \Lambda U \to \Lambda Y$ be rational $\Lambda K$-linear maps. Then $gh$ is $\sigma$-detectable if and only if the pole degrees satisfy $\rho(gh) = \rho(g) + \rho(h)$.*

Consider for a moment three $\Lambda K$-linear maps $f, g, h$ for which the composition $fgh$ is meaningful. Denoting $l_1 := fg$ and $l := gh$, it might, of course, happen that $l_1 h$ is $\sigma$-detectable, whereas $fl_2$ is not. Thus, $\sigma$-detectability is a *non-associative* property. In view of this observation, we shall always use parentheses, and, when saying that $(fg)h$ is $\sigma$-detectable, we shall mean that so is $l_1 h$.

We next examine the internal stability of the configuration shown in the Figure, using the notation introduced there. We recall that a composite system is internally stable whenever all its modes, including the unreachable and the unobservable ones, are stable. First we note that one can construct $f_{(v,r)}$ in two steps: (i) Combine $f$ and $v$ into $g := fv$, and (ii) close the feedback loop through $r$ around $g$. Now, unstable hidden modes can be generated either in step (i) or in step (ii). In step (i), we have just seen that $g$ contains no hidden unstable modes if and only if $fv$ is $\sigma$-detectable. Regarding step (ii), we denote

$$l_r := [I + rg]^{-1} = [I + rfv]^{-1}$$

and $g_r := gl_r$, so that $g_r = f_{(v,r)}$. Then, it was shown in Hammer (1983 a) that the feedback loop (by itself) is internally stable if and only if all of $g_r$, $l_r$, $g_r r$ and $l_r r$ are i/o stable. Combining these two steps, we obtain the following proposition.

*Proposition* 4.3

*Let $f: \Lambda U \to \Lambda Y$ be a linear i/o map, and let $v: \Lambda U \to \Lambda U$ and $r: \Lambda Y \to \Lambda U$, where $v$ is non-singular, be causal $\Lambda K$-linear maps. Then, $f_{(v,r)}$ is internally stable if and only if* (i) *$fv$ is $\sigma$-detectable, and* (ii) *all of $f_{(v,r)}$, $l_r$, $f_{(v,r)} r$ and $l_r r$ are i/o stable.*

In intuitive terms, we next show that internal stability can be characterized as a situation in which the unstable zeros of $f_{(v,r)}$ are the collection of the unstable zeros of $f$, the unstable zeros of $v$, and the unstable poles of the feedback $r$. In technical terms, this fact is reflected in the following two statements, the first of which follows by Hammer and Khargonekar (1981, Proposition 2.10).

*Lemma* 4.1

*Let $f : \Lambda U \to \Lambda Y$ be a linear i/o map, and let $v : \Lambda U \to \Lambda U$ and $r : \Lambda Y \to \Lambda U$, where $v$ is non-singular, be causal and rational $\Lambda K$-linear maps. Also, let $fv = ND^{-1}$ and $r = Q^{-1}R$ be (right and left) canonical stability representations, and denote $G := l_r^{-1} D$. Then $f_{(v,r)}$ is internally stable if and only if* (i) *$fv$ is $\sigma$-detectable, and* (ii) *$QG$ is $\Omega_\sigma K$-unimodular.*

*Corollary* 4.1

*Let $f : \Lambda U \to \Lambda Y$ be a linear i/o map, and let $v : \Lambda U \to \Lambda U$ and $r : \Lambda Y \to \Lambda U$, where $v$ is non-singular, be causal rational $\Lambda K$-linear maps. Assume that $f_{(v,r)}$ is internally stable. Then, the following relations between zero ($\zeta(\cdot)$) and pole ($\rho(\cdot)$) degrees hold:*

(i) $\zeta(l_{(v,r)}) = \rho(f) + \zeta(v) + \rho(r)$

(ii) $\zeta(f_{(v,r)}) = \zeta(f) + \zeta(v) + \rho(r)$

*Proof of Corollary* 4.1

We tacitly employ Proposition 4.3. First we show that $l_{(v,r)}$ is i/o stable. Let $v = Q'P^{-1}$ and $fv = ND^{-1}$ be right canonical stability representations. Then, since $fv$ is $\sigma$-detectable, it follows by Proposition 4.1 that $P$ is a left $\sigma^+$-divisor of $D$. But then, since by Lemma 4.1 $G^{-1} := D^{-1}l_r$ is i/o stable, so also is $P^{-1}l_r$, and hence also $vl_r(= l_{(v,r)})$. Now, since both $l_r$ and $vl_r$ are i/o stable, we have $\zeta(vl_r) = \zeta(v) + \zeta(l_r) - \rho(v)$. Further, using Lemma 4.1 and its notation, we have that $l_r = DG^{-1}$, and $\zeta(l_r) = \zeta(D) + \zeta(G^{-1}) = \zeta(D) + \zeta(Q) = \rho(fv) + \rho(r) = \rho(f) + \rho(v) + \rho(r)$, where, in the last step, we used Proposition 4.2. Thus, we finally obtain that $\zeta(l_{(v,r)})(= \zeta(vl_r)) = \rho(f) + \zeta(v) + \rho(r)$, which proves (i). The proof of (ii) is similar. □

The previous proof also contains the following corollary, which states that a necessary condition for the existence of a representation $fl \overset{\sigma}{=} f_{(v,r)}$ is that $l$ can be chosen i/o stable.

*Corollary* 4.2

*Let $f : \Lambda U \to \Lambda Y$ be a linear i/o map, and let $v : \Lambda U \to \Lambda U$ and $r : \Lambda Y \to \Lambda U$, where $v$ is non-singular, be causal and rational $\Lambda K$-linear maps. If $f_{(v,r)}$ is internally stable, then $l_{(v,r)}$ is i/o stable.*

The conditions for internal stability can also be stated in terms of $\sigma$-detectability: no unstable cancellations in the triple composition $rfv$, as follows.

*Corollary* 4.3

*Let $f : \Lambda U \to \Lambda Y$ be a linear i/o map, and let $v : \Lambda U \to \Lambda U$ and $r : \Lambda Y \to \Lambda U$, where $v$ is non-singular, be causal rational $\Lambda K$-linear maps. Then, $f_{(v,r)}$ is internally stable if and only if* (i) *$fv$ as well as $r(fv)$ are $\sigma$-detectable, and* (ii) *$l_r$ is i/o stable.*

*Proof of Corollary* 4.3

We use the notation of Lemma 4.1. Assume first that $f_{(v,r)}$ is internally stable. Then, by Proposition 4.3, $fv$ is $\sigma$-detectable and $l_r$ is i/o stable. Further, from Lemma 4.1, we have that $[QD+RN](=QG)$ is $\Omega_\sigma K$-unimodular. But then, it follows that $Q$ and $RN$ are left-, whereas $D$ and $RN$ are right-$\sigma^+$-coprime, so that, by Proposition 4.1, $r(fv)$ is $\sigma$-detectable. Conversely, assume that $fv$ and $r(fv)$ are $\sigma$-detectable, and that $l_r$ is i/o stable. Then, by Proposition 4.1, the fact that $r(fv)$ is $\sigma$-detectable implies that $Q$ and $(QD+RN)$ are left-, whereas $D$ and $(QD+RN)$ are right-$\sigma^+$-coprime. But then, since $l_r = D[QD+RN]^{-1}Q$ is i/o stable, it follows that $(QD+RN)^{-1}$ is i/o stable as well. Thus, $(QD+RN)$ is $\Omega_\sigma K$-unimodular, and $f_{(v,r)}$ is internally stable by Lemma 4.1. □

In the remaining part of this section we consider injective systems. First, we show that the case of injective systems can be reduced to the case of isomorphic systems. To this end, let $f: \Lambda U \to \Lambda Y$ be an injective linear i/o map. In view of Lemma 2.2 and the injectivity of $f$, there exists an $\Omega_\sigma{}^-K$-unimodular map $l_0 : \Lambda Y \to \Lambda Y$ such that $l_0 f = \begin{pmatrix} f^0 \\ 0 \end{pmatrix}$, where $f^0 : \Lambda U \to \Lambda U$ is an isomorphism. Further, let $r' : \Lambda Y \to \Lambda U$ be a causal and rational $\Lambda K$-linear map, where $f_{(v,r')}$ is not necessarily internally stable. Define

$$\left.\begin{aligned} r_0 &:= r' f (f^0)^{-1} : \Lambda U \to \Lambda U \\ r &:= (r_0, 0) l_0 : \Lambda Y \to \Lambda U \end{aligned}\right\} \tag{11}$$

Then $rf = r_0 f^0 = r'f$, and, since $l_0$ is $\Omega_\sigma{}^-K$-unimodular, it also follows that $r_0$ and $r$ are both causal.

Now, given a non-singular, causal and rational $\Lambda K$-linear map $v : \Lambda U \to \Lambda U$, we denote, as before, $f^0{}_{(v,r_0)} := f^0 v[I + r_0 f^0 v]^{-1}$, so that we obtain

$$f_{(v,r')} = l_0^{-1} \begin{pmatrix} f^0{}_{(v,r_0)} \\ 0 \end{pmatrix} \tag{12}$$

and

$$f_{(v,r')} = f_{(v,r)} \tag{13}$$

Moreover, using again the fact that $l_0$ is $\Omega_\sigma{}^-K$-unimodular, one can show, through an argument similar to the one used in the last section of Hammer (1983), that the following holds.

*Theorem* 4.1

*Let $f : \Lambda U \to \Lambda Y$ be an injective linear i/o map, and let $v : \Lambda U \to \Lambda U$ and $r' : \Lambda Y \to \Lambda U$, where $v$ is non-singular, be causal and rational $\Lambda K$-linear maps. Then, in the notation of* (11), *the following are equivalent :*

(i) *There exists a causal map $r_* : \Lambda Y \to \Lambda U$ such that $f_{(v,r')} \stackrel{\sigma}{=} f_{(v,r_*)}$.*

(ii) *$f_{(v,r)}$ is internally stable.*

(iii) *$f^0{}_{(r,r_0)}$ is internally stable.*

Thus, one can always implement $r_* = r$, without disturbing internal stability.

## 5. Feedback representation

The present section is devoted to the construction of compensators $w$, $v$ and $r$, as required in the feedback representation problem (§ 1). It will be convenient to start our discussion with the case of injective systems, and to generalize it later to the non-injective case (see the end of the section). To set up our notation, let $f: \Lambda U \rightarrow \Lambda Y$ be an injective linear i/o map, and let $l: \Lambda U \rightarrow \Lambda U$ be a non-singular and causal precompensator. Let $D_\sigma$ be a $\sigma$-annihilator of $f$, and let $D_0$ be a right $\sigma^+$-denominator of $fD_\sigma^{-1}$. Now, assume that there exists a representation $fl \overset{\sigma}{=} f_{(v,r)}$. Then $l = v[I + rfv]^{-1}$; the map $fl$ is evidently i/o stable; and, in view of Corollary 4.2, $l$ is i/o stable as well. For reasons that will become clear shortly, we are interested in the multiple $(D_\sigma l)$ of the stable matrices $D_\sigma$ and $l$, which clearly satisfies

$$(D_\sigma l)^{-1} = v^{-1}D_\sigma^{-1} + rfD_\sigma^{-1} \tag{14}$$

Letting $PA^{-1}$ and $QB^{-1}$ be right canonical stability representations of $v^{-1}D_\sigma^{-1}$ and $rfD_\sigma^{-1}$, respectively, we obtain a reduced partial-fraction decomposition

$$(D_\sigma l)^{-1} = PA^{-1} + QB^{-1} \tag{15}$$

This partial fraction decomposition is at the centre of our discussion, and it actually identifies the main step in our solution of the feedback representation problem. The fact that $f_{(v,r)}$ is internally stable implies that (15) has certain particular properties, which we now proceed to examine.

($\alpha$) *$A$ and $B$ are left $\sigma^+$-coprime, and have $D_\sigma l$ as a $\sigma^+$-LCRM*

*Proof*

By Proposition 3.2, our proof will conclude upon showing that $\zeta(D_\sigma l) = \zeta(A) + \zeta(B)$. Now, since always $\zeta(D_\sigma l) \leqslant \zeta(A) + \zeta(B)$, we only have to prove the converse inequality. Evidently, $\zeta(D_\sigma l) = \zeta(l) + \zeta(D_\sigma)$, so that, by Corollary 4.1, $\zeta(D_\sigma l) = \rho(f) + \zeta(v) + \rho(r) + \zeta(D_\sigma)$. Also,

$$\zeta(A) = \rho(v^{-1}D_\sigma^{-1}) \leqslant \rho(v^{-1}) + \rho(D_\sigma^{-1}) = \zeta(v) + \zeta(D_\sigma)$$

Finally, since $\rho(fD_\sigma^{-1}) = \rho(f)$, we have $\zeta(B) = \rho(rfD_\sigma^{-1}) \leqslant \rho(r) + \rho(f)$. Thus, $\zeta(B) + \zeta(A) \leqslant \rho(r) + \rho(f) + \zeta(v) + \zeta(D_\sigma) = \zeta(D_\sigma l)$, and our proof concludes. We have also obtained

$$\zeta(D_\sigma l) = \zeta(A) + \zeta(B) = \rho(v^{-1}D_\sigma^{-1}) + \rho(rfD_\sigma^{-1}) \qquad \square$$

($\beta$) *$D_\sigma$ is a left $\sigma^+$-divisor of $A$*

*Proof*

Since $f_{(v,r)}$ is internally stable, we have, by Proposition 4.3, that $fv$ is $\sigma$-detectable. In view of (6), this implies that $D_\sigma v$ is $\sigma$-detectable as well. But then, by Proposition 4.1, we directly obtain ($\beta$). $\square$

($\gamma$) *$D_0$ is a left $\sigma^+$-divisor of $B$*

*Proof*

Since, as shown in the proof of ($\beta$), $D_\sigma v$ is $\sigma$-detectable, it follows by Proposition 4.2 that $\zeta(D_\sigma v) = \zeta(D_\sigma) + \zeta(v)$, and, by the proof of ($\alpha$), we also have that $\zeta(D_\sigma l)(= \zeta(D_\sigma) + \zeta(l)) = \zeta(D_\sigma v) + \rho(rfD_\sigma^{-1})$. Hence

$$\rho(rfD_\sigma^{-1}) = \zeta(D_\sigma l) - \zeta(D_\sigma v) = \zeta(l) - \zeta(v) = \rho(f) + \rho(r)$$

where the last step is by Corollary 4.1. But then, since $\rho(fD_\sigma^{-1})=\rho(f)$, we have $\rho(rfD_\sigma^{-1})=\rho(fD_\sigma^{-1})+\rho(r)$, which, by Propositions 4.1 and 4.2, implies $(\gamma)$. □

We summarize the above in the following proposition.

*Proposition* 5.1

*If $f_{(v,r)}$ is internally stable, then the reduced partial-fraction decomposition* (15) *satisfies conditions* $(\alpha)$, $(\beta)$ *and* $(\gamma)$.

The importance of conditions $(\alpha)$, $(\beta)$ and $(\gamma)$ stems from the fact that they are also sufficient for the construction of compensators $v$ and $r$ satisfying $fl\overset{\sigma}{=}f_{(v,r)}$, as we next discuss. As before, we let $f: \Lambda U\to\Lambda Y$ be an injective linear i/o map with $\sigma$-annihilator $D_\sigma$, and we let $D_0$ be a right $\sigma^+$-denominator of $fD_\sigma^{-1}$. Now, let $l: \Lambda U\to\Lambda U$ be a non-singular, causal, and i/o stable precompensator, and assume that there exists a reduced partial-fraction decomposition

$$(D_\sigma l)^{-1}=PA^{-1}+QB^{-1} \tag{16}$$

which satisfies conditions $(\alpha)$, $(\beta)$ and $(\gamma)$. We construct the maps

$$\left.\begin{aligned} g &:= PA^{-1}+\mathbf{L}^+(QB^{-1}): \Lambda U\to\Lambda U \\ h &:= QB^{-1}-\mathbf{L}^+(QB^{-1}): \Lambda U\to\Lambda U \end{aligned}\right\} \tag{17}$$

where $\mathbf{L}^+$ is the polynomial truncation operator (8). Clearly, $l^{-1}=gD_\sigma+hD_\sigma$, and, since $h$ is causal, we have by Corollary 2.1 that $hD_\sigma$ is strictly causal. In view of Proposition 3.1, it follows then that $gD_\sigma$ is non-singular, and that the map

$$v := (gD_\sigma)^{-1}: \Lambda U\to\Lambda U \tag{18}$$

is causal. Finally, let $l_0: \Lambda Y\to\Lambda Y$ be an $\Omega_\sigma^-K$-unimodular map for which $l_0f=\begin{pmatrix} f^0 \\ 0\end{pmatrix}$, where $f^0: \Lambda U\to\Lambda U$ is an isomorphism, and define the matrices

$$\left.\begin{aligned} r_0 &:= h(f^0D_\sigma^{-1})^{-1}: \Lambda U\to\Lambda U \\ r &:= (r_0, 0)l_0: \Lambda Y\to\Lambda U \end{aligned}\right\} \tag{19}$$

where 0 denotes the $m\times(p-m)$ zero matrix. Since $h$ is causal and $f^0D_\sigma^{-1}$ is bicausal (see proof of Proposition 2.3), we have that $r$ is causal. We next show that the causal maps $v$ and $r$ satisfy $fl\overset{\sigma}{=}f_{(v,r)}$.

*Theorem* 5.1

*Let $f: \Lambda U\to\Lambda Y$ be an injective linear i/o map with $\sigma$-annihilator $D_\sigma$, and let $D_0$ be a right $\sigma^+$-denominator of $fD_\sigma^{-1}$. Let $l: \Lambda U\to\Lambda U$ be a non-singular, causal, and i/o stable precompensator, and assume that there exists a reduced partial-fraction decomposition* (16) *satisfying* $(\alpha)$, $(\beta)$ *and* $(\gamma)$. *Let $v$ and $r$ be the causal maps given by* (18) *and* (19), *respectively. Then, $fl\overset{\sigma}{=}f_{(v,r)}$.*

Before proving this theorem, we give an example for the construction of $v$ and $r$.

*Example*

We let $K$ be the field of real numbers, and let $\sigma$ be the set of all polynomials having their roots in the left half of the complex plane. Let $f = (z-1)/[(z-2)^2(z+1)]$, and assume that the desired transfer matrix is $f' = (z-1)/(z+1)^3$. The equivalent precompensator $l$ is then given by $l = f'/f = (z-2)^2/(z+1)^2$, and it is non-singular, causal and i/o stable, as necessary. Now, one can choose

$$D_\sigma = (z-1)/(z+1)^3, \quad D_0 = (z-2)^2$$

Using standard methods, we construct the reduced partial-fraction decomposition

$$(D_\sigma l)^{-1} = \frac{(z+1)^5}{(z-1)(z-2)^2} = \frac{(z+1)^5}{(z-1)} - \frac{(z-3)(z+1)^5}{(z-2)^2}$$

which clearly satisfies ($\alpha$), ($\beta$) and ($\gamma$). From (17) we obtain

$$g = \frac{(z+1)^5}{z-1} - (z^4 + 6z^3 + 15z^2 + 16z) = \frac{z^3 + 9z^2 + 21z + 1}{z-1}$$

$$h = \frac{21z^2 + 78z + 3}{(z-2)^2}$$

Applying finally (18) and (19)

$$v = (gD_\sigma)^{-1} = \frac{(z+1)^3}{z^3 + 9z^2 + 21z + 1}, \quad r = h(fD_\sigma{}^{-1})^{-1} = \frac{21z^2 + 78z + 3}{(z+1)^2}$$

In view of Theorem 5.1, we have that $f'$ $(= fl) \overset{\sigma}{=} f_{(v,r)}$. We note that all other possible partial-fraction decompositions will be of the form $(D_\sigma l)^{-1} = a/[b(z-1)] + c/[d(z-2)^2]$, where $b$ and $d$ are polynomials with stable roots. □

*Proof of Theorem* 5.1

We use the above notation. By Theorem 4.1 our proof will conclude upon showing that $f^0{}_{(v,r_0)}$ is internally stable. The latter follows by Corollary 4.3 through steps ($a$), ($b$) and ($c$) below. Before proceeding, we note several facts. By ($\beta$) and ($\gamma$), there exist i/o stable maps $A_1$, $B_1 : \Lambda U \to \Lambda U$ such that $A = D_\sigma A_1$ and $B = D_0 B_1$. Also, by the definition of $f^0$, $D_\sigma$ is still a $\sigma$-annihilator of $f^0$, and $D_0$ is a right $\sigma^+$-denominator of $f^0 D_\sigma{}^{-1}$. Whence, the map $N_0 := f^0 D_\sigma{}^{-1} D_0$ is $\Omega_\sigma K$-unimodular. By construction, there exist i/o stable matrices $R$ and $T$ such that $g = RA^{-1}$ and $h = TB^{-1}$ are canonical stability representations. We have then that $r_0 = h(f^0 D_\sigma{}^{-1})^{-1} = TB_1{}^{-1} N_0{}^{-1}$, so that, since $T$ and $B$ are right $\sigma^+$-coprime, $\rho(r_0) = \zeta(B_1)$.

($a$) *$f^0 v$ is $\sigma$-detectable*

We have that $v = A_1 R^{-1}$, so that $f^0 v = (N_0 D_0{}^{-1} D_\sigma)(A_1 R^{-1}) = N_0 D_0{}^{-1} AR^{-1}$. Now, $A$ and $R$ are right $\sigma^+$-coprime by construction, whereas $D_0$ and $A$ are left $\sigma^+$-coprime by ($\alpha$) and ($\gamma$). Thus, since $N_0$ is $\Omega_\sigma K$-unimodular, ($a$) follows by Proposition 5.1. As a consequence, we also obtain then that $\rho(f^0 v) = \zeta(D_0) + \zeta(R)$.

*(b) $r_0(f^0v)$ is σ-detectable*

First, let $A'B'^{-1}$ be a right canonical stability representation of $B^{-1}A$. Then, $(D_\sigma l)^{-1} = RA^{-1} + TB^{-1} = (RB' + TA')(AB')^{-1}$, and, since $AB' = BA'$ is clearly a $\sigma^+$-LCRM of $A$ and $B$, it follows by (α) that $(RB' + TA')$ is $\Omega_\sigma K$-unimodular. Whence, $RB'$ and $TA'$ are right $\sigma^+$-coprime, and, since $r_0 f^0 v = TB_1^{-1} D_0^{-1} AR^{-1} = TB^{-1}AR^{-1} = (TA')(RB')^{-1}$, we obtain $\rho(r_0 f^0 v) = \zeta(RB') = \zeta(R) + \zeta(B')$. Now, since $A$ and $B$ are left $\sigma^+$-coprime, $\zeta(B') = \zeta(B) = \zeta(D_0) + \zeta(B_1)$, and, recalling from above, $\rho(f^0v) = \zeta(D_0) + \zeta(R)$ and $\rho(r_0) = \zeta(B_1)$. Thus,

$$\rho(r_0 f^0 v) = \zeta(R) + \zeta(B') = (\zeta(R) + \zeta(D_0)) + \zeta(B_1) = \rho(f^0 v) + \rho(r_0)$$

and (*b*) follows by Proposition 4.2.

*(c) $l_{r0}$ is i/o stable*

We have $l_{r_0} = v^{-1}l = (D_\sigma v)^{-1}(D_\sigma l) = RA^{-1}(D_\sigma l)$. Now, since $D_\sigma l$ is a $\sigma^+$-LCRM of $A$ and $B$, the matrix $A$ is clearly a left $\sigma^+$-divisor of $D_\sigma l$, whence $l_{r_0}$ is i/o stable. □

Combining Proposition 5.1 and Theorem 5.1, we directly obtain the following.

*Theorem* 5.2

*Let $f : \Lambda U \to \Lambda Y$ be an injective linear i/o map, and let $l : \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator. Let $D_\sigma$ be a σ-annihilator of $f$, and let $D_0$ be a right $\sigma^+$-denominator of $fD_\sigma^{-1}$. Then, the following are equivalent :*

(i) *There exists a pair of causal maps $v : \Lambda U \to \Lambda U$ and $r : \Lambda Y \to \Lambda U$ such that $fl \overset{\sigma}{=} f_{(v,r)}$.*

(ii) *There exists a reduced partial-fraction decomposition $(D_\sigma l)^{-1} = PA^{-1} + QB^{-1}$ satisfying* (α), (β) *and* (γ).

From Theorem 5.2 and the discussion leading to it we see that the set of partial-fraction decompositions satisfying (ii) characterizes the set of all pairs $v$, $r$ for which $fl \overset{\sigma}{=} f_{(v,r)}$. The conditions for the existence of a partial-fraction decomposition satisfying (ii) of Theorem 5.2 can be stated in compact form as a certain factorization condition on the matrix $(D_\sigma l)$, as we next discuss. First we review a definition. An equation of the form $AB' = BA'$, where all maps are i/o stable ; $A$ and $B$ are left $\sigma^+$-coprime ; and $A'$ and $B'$ are right $\sigma^+$-coprime, is called an *interchange equation.* The multiple $AB'$ (or $BA'$) is then said to be *interchangeable.* In these terms, we can restate condition (ii) as follows.

*Proposition* 5.2

*Condition* (ii) *of Theorem* 5.2 *is equivalent to the following* (iii) *:*

(iii) *There exist factorizations $D_\sigma l = AB'$ and $D_\sigma l = BA'$, where all of $A$, $A'$, $B$, $B'$ : $\Lambda U \to \Lambda U$ are i/o stable, and where (a) $AB' = BA'$ is an interchange equation, (b) $D_\sigma$ is a left $\sigma^+$-divisor of $A$, and (c) $D_0$ is a left $\sigma^+$-divisor of $B$.*

*Proof of Proposition* 5.2

If (ii) holds, then, by ($\alpha$), the matrices $A$ and $B$ are left $\sigma^+$-coprime, and there exist i/o stable and right $\sigma^+$-coprime matrices $A'$ and $B'$ satisfying $D_\sigma l = AB' = BA'$. But then, $AB' = BA'$ is an interchange equation, and conditions ($b$) and ($c$) coincide with ($\beta$) and ($\gamma$), respectively. Conversely, assume that (iii) holds. Then, since $A'$, $B'$ are right $\sigma^+$-coprime, $D_\sigma l$ is a $\sigma^+$LCRM of $A$ and $B$, and there exist i/o stable matrices $P, Q$ such that $PB' + QA' = I$. But then

$$(D_\sigma l)^{-1} = (AB')^{-1} = (PB' + QA')(AB')^{-1} = PA^{-1} + QA'B'^{-1}A^{-1} = PA^{-1} + QB^{-1}$$

In view of Proposition 3.3(iii), this partial-fraction decomposition is reduced, and, by our assumptions, it evidently satisfies Theorem 5.2(ii). □

Interchange equations have been previously encountered in several situations in the literature on internally stable linear control. These situations include the so-called ' output regulation problem ' (Wolovich and Ferreira 1979), and the problem of decoupling by dynamic output feedback (Hammer and Khargonekar 1981). However, the problem that we presently face is significantly different from the one previously encountered. In the previous investigations, one had to determine whether a given multiple, say $AB'$, is interchangeable. Solutions to this problem were proposed in Wolovich (1978), and the references cited there. In our present case, however, one is required to find a *factorization* of a given map $(D_\sigma l)$ into an interchangeable multiple, the factors of which have to satisfy certain divisibility conditions. From our present discussion we also see that there is a close relationship between interchange equations, and partial-fraction decomposition of matrices.

Let $f : \Lambda U \to \Lambda Y$ be an injective linear i/o map. Using Proposition 5.2 we show now that a partial-fraction decomposition satisfying Theorem 5.1(ii) exists for almost every non-singular, causal and i/o stable precompensator $l : \Lambda U \to \Lambda U$ for which $fl$ is i/o stable. Let $D$ be a right pole matrix of $f$, let $Z$ be a right zero matrix of $f$, and let $Z_l{}^f$ be a right zero matrix of $D^{-1}l$. (We note that $Z_l{}^f$ describes the unstable zeros of $l$ which are not cancelled by unstable poles of $f$.) We denote by $\mathbf{Z}_f$ (respectively, by $\mathbf{P}_f$, $\mathbf{Z}_l{}^f$) the set of all polynomial prime divisors of the invariant factors of $Z$ (respectively, $D$, $Z_l{}^f$). The symbols $\mathbf{Z}_f$, $\mathbf{P}_f$ and $\mathbf{Z}_l{}^f$ determine the respective unstable zeros or poles. We denote

$$\mathbf{Z}_{f,l} := \emptyset \quad \text{if either} \quad \mathbf{P}_f \cap \mathbf{Z}_l{}^f = \emptyset \quad \text{or} \quad \mathbf{Z}_f \cap \mathbf{Z}_l{}^f = \emptyset$$

where $\emptyset$ is the empty set. When the field $K$ is infinite, then we clearly have $\mathbf{Z}_{f,l} = \emptyset$ for almost every precompensator $l$.

*Proposition* 5.3

*Let $f : \Lambda U \to \Lambda Y$ be an injective linear i/o map, and let $l : \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator for which $fl$ is i/o stable. If $\mathbf{Z}_{f,l} = \emptyset$, then there exists a pair of causal maps $v : \Lambda U \to \Lambda U$ and $r : \Lambda Y \to \Lambda U$ satisfying $fl \overset{\sigma}{=} f_{(v,r)}$.*

The proof of Proposition 5.3 depends on the following.

*Lemma* 5.1

*Let $A$, $B$ : $\Lambda U \rightarrow \Lambda U$ be non-singular i/o stable maps. If* det $A$ *and* det $B$ *are $\sigma^+$-coprime, then the multiple $AB$ is interchangeable.*

*Proof of Lemma* 5.1

Let $M_1, M_2 : \Lambda U \rightarrow \Lambda U$ be $\Omega_\sigma K$-unimodular matrices such that $\delta := M_1(AB)M_2$ is in Smith canonical form, say $\delta = \mathrm{diag}\,(\delta_1, \ldots, \delta_m)$. For all $i = 1, \ldots, m$, we can clearly factor $\delta_i = \delta'_i\, \delta''_i$, where $\delta'_i, \delta''_i \in \Omega_\sigma K$ ; $\delta'_i$ is a $\sigma^+$-divisor of det $B$ ; and $\delta''_i$ is a $\sigma^+$-divisor of det $A$. Denoting $B' := M_1[\mathrm{diag}\,(\delta'_1, \ldots, \delta'_m)]$ and $A' := [\mathrm{diag}\,(\delta''_1, \ldots, \delta''_m)]M_2$, and recalling that det $A$ and det $B$ are $\sigma^+$-coprime, it is easily seen that $AB = B'A'$ is an interchange equation. □

*Proof of Proposition* 5.3

We use the notation of Theorem 5.1 and let $R := D_\sigma l$. Now, clearly, $D_\sigma$ is a left $\sigma^+$-divisor of $R$, and, since $(fD_\sigma^{-1})R = fl$ is i/o stable, we also have that $D_0$ is a left $\sigma^+$-divisor of $R$. Whence, the $\sigma^+$-LCRM $Q$ of $D_0$ and $D_\sigma$ is a left $\sigma^+$-divisor of $R$. To express $Q$, let $CD^{-1}$ be a right canonical stability representation of $D_0^{-1}\, D_\sigma$. Then we can choose $Q = D_0 C = D_\sigma D$, so that there exists an i/o stable map $E$ for which $D_\sigma l = D_0 CE = D_\sigma DE$. Furthermore, it is readily seen that $\mathbf{Z}_f = \mathbf{Z}_{D_\sigma} = \mathbf{Z}_C$ ; $\mathbf{P}_f = \mathbf{Z}_{D_0} = \mathbf{Z}_D$ ; and $\mathbf{Z}_l^f = \mathbf{Z}_E$. Whence, the assumption $\mathbf{Z}_{f,\,l} = \emptyset$ implies that either (i) det $E$ and det $D$ are $\sigma^+$-coprime, or (ii) det $E$ and det $C$ are $\sigma^+$-coprime. By Lemma 5.1 it follows then that in case (i) there exists an interchange equation $DE = E'D'$, whereas in case (ii) there exists an interchange equation $CE = E''C'$. But then, in case (i) the maps $A := D_\sigma E'$, $B := D_0$, $A' := CE$, $B' := D'$ satisfy Proposition 5.2, whereas in case (ii) the maps $A := D_\sigma$, $B := D_0 E''$, $A' := C'$, $B' := DE$ satisfy Proposition 5.2. This completes our proof. □

*Construction*

In case the condition $\mathbf{Z}_l^f = \emptyset$ is satisfied, compensators $v$ and $r$ for which $fl \overset{\sigma}{=} f_{(v,r)}$ are constructed as follows. First, one derives the matrices $A$, $B$, $A'$ and $B'$ described in the proof of Proposition 5.1, using the construction given in the proof of Lemma 5.1. From these matrices, one constructs a reduced partial-fraction decomposition $(D_\sigma l)^{-1} = PA^{-1} + QB^{-1}$ as described in the proof of Proposition 5.2. Then, finally, the compensators $v$ and $r$ are obtained from (18) and (19). □

From Proposition 5.3 we directly obtain the following qualitative.

*Corollary* 5.1

*Assume that the field $K$ is infinite, and let $f$ : $\Lambda U \rightarrow \Lambda Y$ be an injective linear i/o map. Then, for almost every non-singular, causal and i/o stable precompensator $l$ : $\Lambda U \rightarrow \Lambda U$ for which $fl$ is i/o stable, there exists a pair of causal maps $v$ : $\Lambda U \rightarrow \Lambda U$ and $r$ : $\Lambda Y \rightarrow \Lambda U$ such that $fl \overset{\sigma}{=} f_{(v,r)}$.*

Several particular cases of Proposition 5.3 are of independent interest : (i) $f$ has no unstable zeros (i.e. $\mathbf{Z}_f = \emptyset$), and (ii) $f$ is i/o stable (i.e. $\mathbf{P}_f = \emptyset$). In these cases evidently $\mathbf{Z}_{f,\,l} = \emptyset$, so that then every non-singular, causal and i/o stable precompensator $l: \Lambda U \to \Lambda U$ for which $fl$ is i/o stable can be represented as $fl \overset{\sigma}{=} f_{(v,r)}$. The case (ii) when the plant $f$ is i/o stable was also considered by Desoer and Chen (1981), by Zames (1981), and by Francis and Vidyasagar (1980). We also note that if $l$ does not add any new unstable zeros (i.e. $\mathbf{Z}_l{}^f = \emptyset$), then it can always be represented as $fl \overset{\sigma}{=} f_{(v,r)}$.

We turn now to the case of more general precompensators which do not necessarily satisfy the conditions of Theorem 5.1. In such case, an external precompensator $w$ is required, and the construction of the triple $w$, $v$ and $r$ is as follows. Let $f: \Lambda U \to \Lambda Y$ be an injective linear i/o map, and let $l: \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator, for which $fl$ is i/o stable. We next split $l$ into a multiple $l = l_1 w$, where $(a)$ both of $l_1$ and $w$ are i/o stable, $(b)$ $fl_1$ is i/o stable, and $(c)$ there exist causal maps $v: \Lambda U \to \Lambda U$ and $r: \Lambda Y \to \Lambda U$ such that $fl_1 = f_{(v,r)}$. These conditions immediately imply then that $fl \overset{\sigma}{=} f_{(w,v,r)}$, where the configuration

$$f_{(w,v,r)} := f_{(v,r)} w$$

is pictorially described in the Figure.

To this end, let $D$ be a right $\sigma^+$-denominator of $f$, and let $A_l$ be a right zero matrix of $(D^{-1}l)$. Since $A_l$ is determined up to a polynomial unimodular left multipler, we can choose $A_l$ with properly independent rows. Then, denoting by $\zeta_i$ the degree of row $i$ in $A_l$, and letting $(z+\alpha)$ be a first-degree polynomial in the stability set $\sigma$, we construct the map

$$w := (\mathrm{diag}\,[(z+\alpha)^{-\zeta_1}, \ldots, (z+\alpha)^{-\zeta_m}])A_l : \Lambda U \to \Lambda U \tag{20}$$

This map is bicausal and i/o stable, and we note that its Macmillan degree

$$\mu(w) = \zeta(l) - \rho(f)$$

Defining now the map

$$l_1 := l w^{-1} \tag{21}$$

we next show that $w$ and $l_1$ satisfy conditions $(a)$, $(b)$ and $(c)$ above.

*Theorem* 5.3

*Let $f: \Lambda U \to \Lambda Y$ be an injective linear i/o map, and let $l: \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator. Assume that $fl$ is i/o stable, and let $w: \Lambda U \to \Lambda U$ be as in* (20). *Then there exist causal maps $v: \Lambda U \to \Lambda U$ and $r: \Lambda Y \to \Lambda U$, where $v$ is non-singular, such that $fl \overset{\sigma}{=} f_{(w,v,r)}$.*

*Proof of Theorem* 5.3

We use the above notation. First we note that, since $fl$ is i/o stable, so also is $D^{-1}l$, so that the map $D^{-1}lA_l^{-1}$ is $\Omega_\sigma K$-unimodular. As a consequence, $D^{-1}l_1$ is $\Omega_\sigma K$-unimodular, and $l_1$ as well as $fl_1$ are i/o stable. Whence, $\mathbf{Z}_{l_1}{}^f = \emptyset$, so that by Proposition 5.3 there exist causal maps $v: \Lambda U \to \Lambda U$ and $r: \Lambda Y \to \Lambda U$ such that $fl_1 \overset{\sigma}{=} f_{(v,r)}$. Also, since $w$ is i/o stable, we have that $fl = fl_1 w \overset{\sigma}{=} f_{(v,r)} w \overset{\sigma}{=} f_{(w,v,r)}$. □

*The non-injective case*

In the next few paragraphs we briefly describe how our discussion can be generalized to the case of non-injective systems. More specifically, we show that the case of non-injective systems can be reduced to the case of injective systems through a certain $\Omega_\sigma{}^-K$-unimodular transformation. We start with the examination of a particularly simple case of non-injective systems.

Let $f: \Lambda U \to \Lambda Y$ be a linear i/o map. We say that $f$ has a *static kernel* if there exists a $K$-linear subspace $U_0 \subset U$ such that $\operatorname{Ker} f = \Lambda U_0$ (Hammer and Heymann 1981). Assume now that $f$ has a non-zero static kernel, and denote $k := \dim_{\Lambda K} \operatorname{Ker} f$ and $n := \dim_K U - k$. Then, choosing a suitable basis for the $K$-linear space $U$, we can assume that the transfer matrix of $f$ is of the form $(f_0, 0)$, where $f_0$ is injective. We denote by $\Lambda U_1$ the domain of $f_0$, so that $f_0: \Lambda U_1 \to \Lambda Y$, and $U = U_1 \oplus U_0$. Now, let $l: \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator for which $fl$ is i/o stable, and let $l^n$ be the matrix consisting of the first (upper) $n$ rows of $l$. Then, since $l$ is non-singular, there exists a non-singular (column interchanging) static matrix $V: U \to U$ such that the first $n$ columns of $l^n V$ are $\Lambda K$-linearly independent. We denote $l' := lV$, and partition $l^n V$ into $l^n V = (l_1, l_2)$, where $l_1: \Lambda U_1 \to \Lambda U_1$ is a (non-singular) $n \times n$ matrix, and $l_2: \Lambda U_0 \to \Lambda U_1$. Next, we define

$$l'' := \begin{pmatrix} l_1 & l_2 \\ 0 & I_{k\times k} \end{pmatrix} : \Lambda U \to \Lambda U \tag{22}$$

where $I_{k\times k}$ is the $k \times k$ identity matrix. Then, we evidently have that $l''$ is non-singular, causal and i/o stable, and moreover

$$fl = fl' V^{-1} = fl'' V^{-1} \tag{23}$$

Thus, we can replace $l$ by $l'' V^{-1}$ without affecting the resulting system.

Next, let $D_0$ be a right $\sigma^+$-denominator of $f_0$, and, referring to (22), let $l_*: \Lambda U_1 \to \Lambda U_1$ be a greatest common left divisor of $l_1$ and $l_2$ over the ring $\Omega_\sigma{}^-K$. We note that, by the structure of the rings $\Omega_\sigma K$ and $\Omega_\sigma{}^-K$, $l_*$ is a greatest common left $\sigma^+$-divisor of $l_1$ and $l_2$ as well. Now, since $fl''$ is i/o stable, so also are both of $D_0{}^{-1} l_1$ and $D_0{}^{-1} l_2$. Hence, $D_0$ is a left $\sigma^+$-divisor of $l_*$, and it follows that $f_0 l_*$ is i/o stable. Further, since $l_1$ is non-singular, so also is $l_*$, and the $\Lambda K$-linear maps $l_\alpha := l_*{}^{-1} l_1$ and $l_\beta := l_*{}^{-1} l_2$ are both causal and i/o stable. We now define the following non-singular, causal and i/o stable maps

$$l_0 := \begin{pmatrix} l_* & 0 \\ 0 & I_{k\times k} \end{pmatrix} : \Lambda U \to \Lambda U$$

$$l_K := \begin{pmatrix} l_\alpha & l_\beta \\ 0 & I_{k\times k} \end{pmatrix} : \Lambda U \to \Lambda U$$

so that $l'' = l_0 l_K$. Our construction implies then (i) $fl_0$ *is i/o stable, and* (ii) $\operatorname{Ker} fl_0 = \operatorname{Ker} f$. Thus, we can consider $fl_0$, and then add $l_K$ as an external precompensator without destroying internal stability.

In order to complete our discussion for the case of static kernels, we now consider the injective combination $f_0 l_*$. Applying Theorem 5.3, we obtain causal $\Lambda K$-linear maps $w_*, v_* : \Lambda U_1 \rightarrow \Lambda U_1$ and $r_* : \Lambda Y \rightarrow \Lambda U_1$, where $w^*$ and $v_*$ are non-singular, such that $f_0 l_* \overset{\sigma}{=} f_{0(w_*, v_*, r_*)}$. Defining now

$$\left.\begin{aligned} w &:= \begin{pmatrix} w^* & 0 \\ 0 & I_{k\times k} \end{pmatrix} l_K V^{-1} : \Lambda U \rightarrow \Lambda U \\ v &:= \begin{pmatrix} v_* & 0 \\ 0 & I_{k\times k} \end{pmatrix} \qquad\quad : \Lambda U \rightarrow \Lambda U \\ r &:= \begin{pmatrix} r_* \\ 0 \end{pmatrix} \qquad\qquad\qquad : \Lambda Y \rightarrow \Lambda U \end{aligned}\right\} \tag{24}$$

it then readily follows that $fl \overset{\sigma}{=} f_{(w,v,r)}$.

Finally, we consider the general case of non-injective systems. Specifically, we show that the general case can be reduced to the case of systems with static kernels. To this end, let $f : \Lambda U \rightarrow \Lambda Y$ be a non-injective linear i/o map, and let $l : \Lambda U \rightarrow \Lambda U$ be a non-singular, causal and i/o stable precompensator, for which $fl$ is i/o stable. We denote, as before, $k := \dim_{\Lambda K} \operatorname{Ker} f$, and assume that $k > 0$. By the dual of Lemma 2.2, there exists an $\Omega_\sigma$-$K$-unimodular map $l_U : \Lambda U \rightarrow \Lambda U$ such that $fl_U = (f_0, 0)$, where $f_0 : \Lambda U_1 \rightarrow \Lambda Y$ is an injective linear i/o map. Evidently, the linear i/o map $f_S := fl_U$ has a static kernel. Further, we denote $l_S := l_U^{-1} l$. Then, clearly, $f_S l_S = fl$; the precompensator $l_S$ is non-singular, causal and i/o stable; and $f_S l_S$ is i/o stable. Thus, we can apply our previous constructions to the pair $f_S$, $l_S$. As in (24), we let $w$, $v$ and $r$ be causal $\Lambda K$-linear maps such that $f_S l_S \overset{\sigma}{=} f_{S(w,v,r)}$. Then, defining $v_U := l_U v$, and using the fact that $l_U$ is $\Omega_\sigma$-$K$-unimodular, it can readily be shown that

$$fl \overset{\sigma}{=} f_{(w, v_U, r)} \tag{25}$$

Finally, we note that if the precompensator $l$ is such that $fl$ has a static kernel, then $l_K$ in (24) is static, and the situation is essentially the same as in the injective case.

## References

Cheng, L., and Pearson, J. B., 1978, *I.E.E.E. Trans. autom. Control*, **23**, 3.
Davison, E. J., 1976, *I.E.E.E. Trans. autom. Control*, **21**, 25.
Desoer, C. A., and Chan, W. S., 1975, *J. Franklin Inst.*, **300**, 335.
Desoer, C. A., and Chen, M. J., 1981, *I.E.E.E. Trans. autom. Control*, **26**, 408.
Desoer, C. A., Liu, R. W., Murray, J., and Saeks, R., 1980, *I.E.E.E. Trans. autom. Control*, **25**, 399.
Desoer, C. A., and Vidyasagar, M., 1975, *Feedback Systems: Input–Output Properties* (New York: Academic Press).
Francis, B. A., and Vidyasagar, M., 1981, S. & I.S. Report 8003, University of Waterloo, Waterloo, Canada.

FUHRMANN, P. A., 1981, *On Generalized Bezoutians*, Preprint, Department of Mathematics, University of the Negev, Beer Sheba, Israel.

HAMMER, J., 1981 a, *Stability and Stable Precompensation: An Algebraic Approach*, Preprint, Center for Mathematical Systems Theory, University of Florida, Gainesville, Florida 32611, U.S.A.; 1981 b, *On Stability, Causality and Dynamic Output Feedback*, *Ibid.*; 1983 a, *I.E.E.E. Trans. autom. Control*, **28** (to appear); 1983 b, *SIAM J. Control Optim.*, **21** (to appear); 1983 c, *Int. J. Control*, **37,** 62.

HAMMER, J., and HEYMANN, M., 1981, *SIAM J. Control Optim.*, **19,** 445; 1983 a, *Ibid.*, **21** (to appear); 1983 b, *Linear Algebra Applic.* (to appear).

HAMMER, J., and KHARGONEKAR, P. P., 1981, *Decoupling of Linear Systems by Dynamic Output Feedback*, Preprint, Center for Mathematical System Theory, University of Florida, Gainesville, FL 32611, U.S.A.

HAUTUS, M. L. J., and HEYMANN, M., 1978, *SIAM J. Control Optim.*, **16,** 83.

KALMAN, R. E., FALB, P. L., and ARBIB, M. A., 1969, *Topics in Mathematical System Theory* (New York: McGraw-Hill).

MACDUFFEE, C. C., 1934, *The Theory of Matrices* (New York: Chelsea).

MORSE, A. S., 1975, *Lecture Notes in Economics and Mathematical Systems*, No. 131 (New York: Springer Verlag), pp. 183–194.

NEWTON, G. C., GOULD, L. A., and KAISER, J. F., 1957, *Analytical Design of Linear Feedback Controls* (New York: Wiley).

PERNEBO, L., 1981, *I.E.E.E. Trans. autom. Control*, **26,** 171.

ROSENBROCK, H. H., 1970, *State Space and Multivariable Theory* (London: Nelson).

ROSENBROCK, H. H., and HAYTON, G. E., 1977, *Int. J. Control*, **27,** 837.

WOLOVICH, W. A., 1974, *Linear Multivariable Systems*, Applied Mathematical Sciences Series No. 11 (New York: Springer Verlag); 1978, *I.E.E.E. Trans. autom. Control*, **20,** 148.

WOLOVICH, W. A., and FERREIRA, P., 1979, *I.E.E.E. Trans. autom. Control*, **24,** 460.

WONHAM, W. M., 1974, *Linear Multivariable Control: A Geometric Approach*, Lecture Notes in Economics and Mathematical Systems, No. 101 (New York: Springer Verlag).

WONHAM, W. M., and PEARSON, J. B., 1974, *SIAM J. Control Optim.*, **12,** 5.

WYMAN, B. F., 1972, *Linear Systems over Commutative Rings*, Lecture Notes, Stanford University, Stanford, CA, U.S.A.

YOULA, D. C., BONGIORNO, J. J., and JABR, H. A., 1976, *I.E.E.E. Trans. autom. Control*, **21,** 319.

ZAMES, G., 1981, *I.E.E.E. Trans. autom. Control*, **26,** 301.