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Fastest recovery after feedback disruption: nonlinear delay-differential systems

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ABSTRACT
The problem of reducing operating errors after a feedback disruption is considered for a class of nonlinear delay-differential systems. It is shown that there are optimal controllers that reduce such errors in minimal time, once feedback has been restored. It is further shown that optimal performance can be approximated by bang-bang controllers – controllers that are easy to design and implement.

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1. Introduction

Delay-differential systems are encountered in many engineering applications, including remotely controlled systems, systems affected by computational delays, and systems with significant reaction times (e.g. Ailon & Gil, 2000; Bushnell, 2001; Imaida, Yokokohji, Doi, Oda, & Yoshikawa, 2004; Sheridan & Ferrell, 1963). The present paper concentrates on the control of delay-differential systems in the aftermath of a feedback disruption. Feedback disruptions are quite common in engineering practice; they may occur as a result of malfunctions; poor operating conditions (e.g. loss of line of sight to a satellite); demands for stealthy operation; need to conserve energy; or overload of feedback communication channels (Montestruque & Antsaklis, 2004; Nair, Fagnani, Zampieri, & Evans, 2007; Zhivogyladov & Middleton, 2003). Feedback disruptions are also an integral aspect of the operation of sampled-data control systems, where no feedback is available between samples.

Basic principles of feedback control suggest that operating errors may increase during periods of feedback disruption. The current paper develops robust controllers that reduce such operating errors in minimal time, once feedback has been restored.

Consider the control configuration of Figure 1. Here, the controlled system Σ is a nonlinear delay-differential system with input signal $u(t)$ and state $x(t)$. The controller C had suffered a disruption in its feedback channel for some time, until feedback was restored momentarily at the time $t = 0$. At this time, C received the sample $x(0) = x_0$ of the state of Σ. Using this sample of the state, C must guide Σ to reduce in minimal time any operating errors that may have accumulated during open-loop operation.

After possibly shifting the state space coordinates of Σ, we assume that nominal operation is at the zero state $x = 0$. The objective of the controller C is to bring Σ to the zero state as quickly as possible. Due to errors and uncertainties prevalent in practice, it is not possible to bring the state of Σ exactly to the zero state. Instead, a deviation of magnitude not exceeding ℓ is allowed. Then, once feedback has been restored, the goal of C is to guide Σ to reduce operating errors as quickly as possible to a level of ℓ or below. We show in Section 4 that optimal controllers fulfilling this goal exist under rather broad conditions. The main requirement for the existence of such controllers is controllability of the system Σ.

Optimal controllers are often difficult to design and implement. In Section 5, we show that, without significantly degrading performance, optimal controllers can be replaced by controllers that generate bang-bang input signals for Σ. Such controllers are relatively easy to design and implement, since bang-bang signals are characterised by a finite list of real numbers – their switching times. All controllers developed in this paper are robust; they can accommodate inaccuracies, modelling errors, and uncertainties.

The current paper expands the work of Yu and Hammer (2016a, 2016b) from systems described by ordinary differential equations to systems described by delay-differential equations. Our discussion depends on the literature on min–max optimisation, including Kelendzheridze (1961), Pontryagin, Boltyansky, Gamkrelidze, and Mishchenko (1962), Neustadt (1966, 1967),

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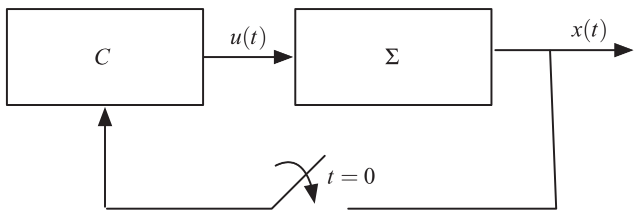


Figure 1. Control system.

Gamkrelidze (1965), Luenberger (1969), Young (1969), Warga (1972), Chakraborty and Hammer (2007, 2008a, 2008b, 2008c, 2009a, 2009b, 2010), Chakraborty and Shaikshavali (2009), the references cited in these works, and many others. A discussion of recent advances in the theory of delay-differential systems can be found in Loiseau, Michiels, Niculescu, and Sipahi (2009), Niculescu and Gu (2012), the references cited in these works, and many others. Yet, it seems that there are no earlier reports in the literature addressing the existence, implementation, or approximation of optimal controllers that reduce operating errors of delay-differential systems.

The paper is organised as follows. Section 2 introduces the basic mathematical framework, and Section 3 presents a few preliminary facts. The existence of optimal controllers is proved in Section 4, while Section 5 shows that optimal performance can be approximated by bang-bang controllers. Section 6 consists of an example, and concluding remarks are given in Section 7.

2. Formal problem statement

2.1 Notation

As usual, R denotes the real numbers; R^+ denotes the non-negative real numbers; R^n is the set of all column vectors with n real components; and $R^{n\times m}$ is the set of all $n \times m$ matrices with real entries. The absolute value of a real number r is $|r|$; the L^∞-norm of a constant matrix $A = (a_{ij}) \in R^{n\times m}$ is $|A| = \max_{i,j}|a_{ij}|$. For a matrix-valued function of time v: $R^+ \to R^{n\times m} : t \mapsto v(t)$, the L^∞ −norm is $|v|_\infty := \sup_{t\geq 0} |v(t)|$, where $|v|_\infty := \infty$ if there is no supremum. The function v is *bounded* if $|v|_\infty < \infty$. We often refer to $|v|_\infty$ as the *amplitude* of v.

It is important to distinguish between the two norms $|v(t)|$ and $|v|_\infty$: the first is the largest absolute value of an entry of $v(t)$ at the time t, while the second is the supremal absolute value of the entries over time.

Due to physical constraints and component limitations, practical systems often impose a bound on the maximal amplitude of their input signal. Thus, the system Σ of Figure 1 imposes an amplitude bound of $K > 0$ on its input signals; only input signals satisfying $|u|_\infty \leq K$ are allowed.

2.2 The class of signals

We use the mathematical framework of Chakraborty and Hammer (2009b, 2010). Let L^m be the linear space of Lebesgue measurable functions f: $R \to R^m : t \mapsto f(t)$ that are zero at all times $t < 0$. For a real number $\omega > 0$, denote by $L_2^{\omega,m}$ the Hilbert space of functions $f, g \in L^m$ with the inner product

$$\langle f, g\rangle := \int_0^\infty e^{-\omega s} f^T(s)g(s)ds, \tag{2.1}$$

where f^T is the transpose of f. Note that the inner product (2.1) is bounded whenever f and g are bounded.

The class of permissible input signals $U(K)$ of the controlled system Σ of Figure 1 consists of all members of $L_2^{\omega,m}$ that are bounded by K, i.e.

$$U(K) := \left\{u \in L_2^{\omega,m} : |u|_\infty \leq K\right\}. \tag{2.2}$$

The following notation is convenient. Let $D(t)$ be an $n \times m$ matrix-valued function of time with the rows $D_1(t), D_2(t), \dots, D_n(t)$, where $D_j^T(t) \in L_2^{\omega,m}$, $j = 1, 2, \dots, n$. Then, for a function $g \in L_2^{\omega,m}$, denote

$$\langle D, g\rangle := \sum_{j=1}^{n} \langle D_j^T, g\rangle. \tag{2.3}$$

2.3 The class of controlled systems

We consider input-affine systems described by delay-differential equations of the form

$$\Sigma : \begin{array}{l} \dot{x}(t) = a(t, x(t), x(t-\tau)) + b(t, x(t), x(t-\tau))w(t) \\ \quad + c(t, x(t), x(t-\tau))w(t-\tau), \\ x(0) = x_0. \end{array} \tag{2.4}$$

Here, $x(t) \in R^n$ is the state and $w(t) \in R^m$ is the input signal; $\tau > 0$ is the time delay; and a: $R \times R^n \times R^n \to R^n$: $(t, y, z) \mapsto a(t, y, z)$, b: $R \times R^n \times R^n \to R^{n\times m}$: $(t, y, z) \mapsto b(t, y, z)$, and c: $R \times R^n \times R^n \to R^{n\times m}$: $(t, y, z) \mapsto c(t, y, z)$ are continuous functions. The initial state $x(0) = x_0$ is known, having been communicated by the feedback when it was restored momentarily at $t = 0$ (see Section 1).

Due to the delay time τ, the state $x(t)$ at times $t > 0$ depends, among other quantities, on values of the input signal $w(t)$ during times $t < 0$. However, the input segment $w(t)$, $t < 0$, is not under our control, since control action starts at $t = 0$. Therefore, we must distinguish among input values received by Σ before and after $t = 0$. We refer to input received before $t = 0$ as the *residual input signal* $v(t)$; there is no control over values of $v(t)$. Input received after $t = 0$ is the *control input signal* $u(t)$, so that

$$w(t) = \begin{cases} v(t) & t < 0, \\ u(t) & t \geq 0. \end{cases} \quad (2.5)$$

Both $v(t)$ and $u(t)$ are Lebesgue measurable functions bounded by K, and so is $w(t)$. The signal v is a remnant of past operating policies; the control input signal $u(t)$ is specifically designed to achieve our control objective of reducing operating errors in minimal time. The design of u is the subject of this paper.

To incorporate modelling uncertainties and errors into our discussion, we decompose the functions a, b, and c of (2.4) into two parts: specified parts a_0, b_0, c_0 that represent the nominal model and unspecified parts $a_\gamma, b_\gamma, c_\gamma$ that represent modelling uncertainties and errors:

$$\begin{aligned} a(t, y, z) &= a_0(t, y, z) + a_\gamma(t, y, z), \\ b(t, y, z) &= b_0(t, y, z) + b_\gamma(t, y, z), \\ c(t, y, z) &= c_0(t, y, z) + c_\gamma(t, y, z). \end{aligned} \quad (2.6)$$

Here, a_0: $R \times R^n \times R^n \to R^n$, b_0: $R \times R^n \times R^n \to R^{n \times m}$, and c_0: $R \times R^n \times R^n \to R^{n \times m}$ are specified continuous functions, while a_γ: $R \times R^n \times R^n \to R^n$, b_γ: $R \times R^n \times R^n \to R^{n \times m}$, and c_γ: $R \times R^n \times R^n \to R^{n \times m}$ are unknown continuous functions. All functions are subject to the following Lipschitz conditions at all times t and for all y, z, y', $z' \in R^n$:

$$\begin{aligned} &\left|a_0(t, y', z') - a_0(t, y, z)\right| \leq \alpha \max\left\{\left|y' - y\right|, \left|z' - z\right|\right\}, \\ &\left|b_0(t, y', z') - b_0(t, y, z)\right| \leq \alpha \max\left\{\left|y' - y\right|, \left|z' - z\right|\right\}, \\ &\left|c_0(t, y', z') - c_0(t, y, z)\right| \leq \alpha \max\left\{\left|y' - y\right|, \left|z' - z\right|\right\}, \\ &\quad a_0(t, 0, 0) = 0, |b_0(t, 0, 0)| \leq \alpha, |c_0(t, 0, 0)| \leq \alpha; \end{aligned} \quad (2.7)$$

$$\begin{aligned} &\left|a_\gamma(t, y', z') - a_\gamma(t, y, z)\right| \leq \gamma \max\left\{\left|y' - y\right|, \left|z' - z\right|\right\}, \\ &\left|b_\gamma(t, y', z') - b_\gamma(t, y, z)\right| \leq \gamma \max\left\{\left|y' - y\right|, \left|z' - z\right|\right\}, \\ &\left|c_\gamma(t, y', z') - c_\gamma(t, y, z)\right| \leq \gamma \max\left\{\left|y' - y\right|, \left|z' - z\right|\right\}, \\ &\quad a_\gamma(t, 0, 0) = 0, \left|b_\gamma(t, 0, 0)\right| \leq \gamma, \left|c_\gamma(t, 0, 0)\right| \leq \gamma, \end{aligned} \quad (2.8)$$

where $\alpha, \gamma > 0$ are specified real numbers. The number γ represents modelling uncertainties and is regarded as small. The nominal system is

$$\Sigma_0: \begin{aligned} &\dot{x}(t) = a_0(t, x(t), x(t-\tau)) + b_0(t, x(t), x(t-\tau))w(t) \\ &\quad + c_0(t, x(t), x(t-\tau))w(t-\tau), \\ &x(0) = x_0. \end{aligned} \quad (2.9)$$

In addition to uncertainty about the model of Σ, there is also uncertainty about the residual input signal v. To describe this uncertainty, let v_0: $R \to R^m$ be a Lebesgue measurable function bounded by K that forms the nominal residual input signal. Let $V_\gamma(v_0)$ be the family of all Lebesgue measurable functions v: $R \to R^m$ that are bounded by K and differ by no more than γ from v_0, namely,

$$V_\gamma(v_0) := \left\{ v : R \to R^m \middle| \begin{aligned} &|v(t) - v_0(t)| \leq && t < 0, \\ &\gamma \text{ and } |v(t)| \leq K, && \\ &v(t) = 0, && t \geq 0. \end{aligned} \right\} \quad (2.10)$$

The actual residual input signal v is not known precisely; it is only known that $v \in V_\gamma(v_0)$. This represents an uncertainty about the residual input signal.

Notation 2.1: Let Σ_0 be the nominal system of (2.9), let v_0 be the nominal residual input signal, and let $\gamma > 0$ be a real number. The family of systems $\mathcal{F}_\gamma(\Sigma_0)$ consists of all systems Σ of the form (2.4), where the functions a, b, and c satisfy (2.6), (2.7), and (2.8), and a_γ, b_γ, and c_γ are unspecified continuous functions. All members of $\mathcal{F}_\gamma(\Sigma_0)$ have the same initial state $x(0) = x_0$; all have received the same unspecified residual input signal $v \in V_\gamma(v_0)$; and all receive the same control input signal $u \in U(K)$. The state of Σ at a time t is denoted by $x(t) = \Sigma(x_0, v, u, t) = \Sigma(x_0, w, t)$.

A stationary state is a state at which a system remains when its input is zero. The following is a direct consequence of (2.4) (2.6), (2.7), and (2.8).

Proposition 2.1: *The zero state is a stationary point of every member of $\mathcal{F}_\gamma(\Sigma_0)$.*

As every practical system had started from rest at some point in the past, we adopt the following.

Convention 2.1: There is an *activation time $t_a < 0$* prior to which all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ were at the zero state and had received zero input. ■

2.4 The control objective

Assuming that the state coordinates of the controlled system Σ have been appropriately shifted, we consider the zero state $x = 0$ as the target state of Σ. The control objective is then to bring Σ as quickly as possible from the initial state x_0 to the vicinity of the zero state. Due to the presence of uncertainties and errors, it is, of course, not possible to bring Σ exactly to the zero state; instead, a tolerable deviation $\ell > 0$ from the zero state is specified. Referring to ℓ as the *operating error bound*, the control objective is to bring Σ as quickly as possible to a state x satisfying $x^T x \leq \ell$, or, equivalently, to a state x within the domain

$$\rho(\ell) := \{x \in R^n | x^T x \leq \ell\}. \quad (2.11)$$

This control objective must be achieved without knowing which member of $\mathcal{F}_\gamma(\Sigma_0)$ is the controlled system Σ, and which member of $V_\gamma(v_0)$ is the residual input signal v.

Consider a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with the residual input signal $v \in V_\gamma(v_o)$ and the control input signal $u \in U(K)$. The shortest time $t(x_0, \Sigma, v, u)$ it takes for the state $x(t) = \Sigma(x_0, v, u, t)$ to go from the initial state x_0 to the domain $\rho(\ell)$ is

$$t(x_0, \Sigma, v, u) = \inf_{t \geq 0} \left\{ \Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t) \leq \ell \right\}, \quad (2.12)$$

where $t(x_0, \Sigma, v, u) = \infty$ if a state $x \in \rho(\ell)$ cannot be reached.

Now, we do not actually know which member of $\mathcal{F}_\gamma(\Sigma_0)$ the system Σ is, nor do we know which member of $V_\gamma(v_0)$ the residual input signal v is. Therefore, the control input signal u must bring the state of every member of $\mathcal{F}_\gamma(\Sigma_0)$ into the set $\rho(\ell)$, irrespective of which residual input signal $v \in V_\gamma(v_0)$ was received. The first time $t(x_0, \gamma, \ell, u)$ at which u can bring every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from the initial state x_0 into $\rho(\ell)$, irrespective of which residual input signal $v \in V_\gamma(v_0)$ was received, is

$$t(x_0, \gamma, \ell, u) = \inf_{t \geq 0} \left\{ \left[\sup_{\substack{\Sigma \in \mathcal{F}_\gamma(\Sigma_0) \\ v \in V_\gamma(v_0)}} \Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t) \right] \leq \ell \right\}; \quad (2.13)$$

here, $t(x_0, \gamma, \ell, u) = \infty$ if u cannot bring every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ into $\rho(\ell)$ for every residual input signal $v \in V_\gamma(v_0)$.

Finally, the shortest time $t^*(x_0, \gamma, \ell)$ within which any control input signal $u \in U(K)$ can bring all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ into $\rho(\ell)$, irrespective of which residual input signal $v \in V_\gamma(v_0)$ was used, is

$$t^*(x_0, \gamma, \ell) = \inf_{u \in U(K)} t(x_0, \gamma, \ell, u); \quad (2.14)$$

here, $t^*(x_0, \gamma, \ell) = \infty$ if the infimum does not exist, i.e., if no permissible control input signal can achieve the control objective.

When $t^*(x_0, \gamma, \ell) < \infty$, we have to answer the following critical question: is there a control input signal $u^*(x_0, \gamma, \ell) \in U(K)$ that achieves the time $t^*(x_0, \gamma, \ell)$ so that

$$t^*(x_0, \gamma, \ell) = t(x_0, \gamma, \ell, u^*(x_0, \gamma, \ell)). \quad (2.15)$$

Recall that the same input signal must be used for all systems $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ and all residual input signals $v \in V_\gamma(v_0)$, since the exact system and the exact residual input signal are not known. We show in Section 4 that an optimal control input signal $u^*(x_0, \gamma, \ell)$ does exist whenever the nominal system Σ_0 is controllable and the uncertainty parameter γ is not excessively large.

An optimal control input signal $u^*(x_0, \gamma, \ell)$ may not be easy to implement. In general, $u^*(x_0, \gamma, \ell)$ is a vector-valued function of time; the calculation and implementation of vector-valued functions of time is, more often than not, a vexing task. An important objective of our discussion is to develop control input signals that achieve close-to-optimal performance, while being relatively easy to calculate and implement. We show in Section 5 that optimal performance can be closely approximated by bang-bang control input signals. Bang-bang signals are piecewise constant signals, whose components switch between the extremal values of K and $-K$. Bang-bang signals are relatively easy to calculate and implement, since they are basically determined by a finite list of real numbers – their switching times. In summary, our discussion centres on the following issues.

Problem 2.1: Let $K, \gamma, \ell > 0$ be real numbers. Let $\mathcal{F}_\gamma(\Sigma_0)$ be the family of systems of Notation 2.1 with the initial condition x_0; let $V_\gamma(v_0)$ be the family of residual input signals of (2.10); and let $\rho(\ell)$ be given by (2.11).

(i) Find under what conditions there exists an optimal control input signal $u^*(x_0, \gamma, \ell)$ that satisfies (2.15).
(ii) If $u^*(x_0, \gamma, \ell)$ exists, find simple-to-calculate-and-implement control input signals that closely approximate optimal performance.

We show in Section 3 that optimal control input signals $u^*(x_0, \gamma, \ell)$ exist under rather broad conditions. The main requirement for the existence of $u^*(x_0, \gamma, \ell)$ is a certain controllability condition called *K-controllability* (see Section 3). One would expect that controllability is involved, since it must be possible to drive every member of $\mathcal{F}_\gamma(\Sigma_0)$ into the vicinity of the origin. Among other factors, K-controllability depends on a favourable relationship between the initial state x_0 and the input amplitude bound K (see Section 3).

Furthermore, we show in Section 3 that it is enough to verify K-controllability for the nominal system Σ_0; K-controllability of the nominal system Σ_0 is sufficient to guarantee the existence of an optimal solution $u^*(x_0, \gamma, \ell)$, as long as the uncertainty parameter γ is not excessively large. Other members of $\mathcal{F}_\gamma(\Sigma_0)$ do not have to be checked individually for K-controllability.

Regarding Problem 2.1*(ii)*, we show in Section 5 that optimal performance can be closely approximated by bang-bang input signals. Bang-bang signals are usually easier to calculate and implement than optimal signals.

3. Underlying facts

3.1 Escape times

An escape time is a finite time at which the response of a system diverges. We show now that systems of the family $\mathcal{F}_\gamma(\Sigma_0)$ have no escape times, namely, their response is bounded at finite times; of course, the response may diverge as $t \to \infty$.

Lemma 3.1: *For every time $T \geq 0$, there is a real number $M(T) \geq 0$ such that $|\Sigma(x_0, v, u, t)| \leq M(T)$ for all $t \in [0, T]$, for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, for all $v \in V_\gamma(v_0)$, and for all $u \in U(K)$.*

Proof: Let Σ be a member of $\mathcal{F}_\gamma(\Sigma_0)$, let $t_1 < t_2$ be two times, let $t \in (t_1, t_2]$, and let w be a Lebesgue measurable input signal bounded by K. Then, by (2.4), we can write

$$\begin{aligned} x(t) \quad &= x(t_1) + \int_{t_1}^{t} \Big[a(s, x(s), x(s-\tau)) \\ &\quad + b(s, x(s), x(s-\tau))w(s) \\ &\quad + c(s, x(s), x(s-\tau))w(s-\tau) \Big] ds. \\ &= x(t_1) + \int_{t_1}^{t} [a(s, x(s), x(s-\tau)) - a(s,0,0)]ds \\ &\quad + \int_{t_1}^{t} \{[b(s, x(s), x(s-\tau)) - b(s,0,0)]w(s) \\ &\qquad + b(s,0,0)w(s)\}ds \\ &\quad + \int_{t_1}^{t} \{[c(s, x(s), x(s-\tau)) - c(s,0,0)]w(s-\tau) \\ &\qquad + c(s,0,0)w(s-\tau)\}ds. \end{aligned}$$

Using (2.6), (2.7), and (2.8), together with the fact that w is bounded by K, we obtain

$$\begin{aligned} |x(t)| \leq\ & |x(t_1)| + (\alpha+\gamma)(t-t_1) \sup_{s\in[t_1-\tau,t]} |x(s)| \\ & + (\alpha+\gamma)(t-t_1) \sup_{s\in[t_1-\tau,t]} |x(s)|K \\ & + (\alpha+\gamma)(t-t_1)K \\ & + (\alpha+\gamma)(t-t_1) \sup_{s\in[t_1-\tau,t]} |x(s)|K \\ & + (\alpha+\gamma)(t-t_1)K. \end{aligned}$$

Then, since $\sup_{s\in[t_1-\tau,t]} |x(s)| \leq \sup_{s\in[t_1-\tau,t_1]} |x(s)| + \sup_{s\in[t_1,t]} |x(s)|$, we get

$$\begin{aligned} \sup_{t\in[t_1,t_2]} |x(t)| \leq\ & |x(t_1)| + 2(\alpha+\gamma)(t_2-t_1)K \\ & + (\alpha+\gamma)(1+2K)(t_2-t_1) \sup_{t\in[t_1-\tau,t_1]} |x(t)| \\ & + (\alpha+\gamma)(1+2K)(t_2-t_1) \sup_{t\in[t_1,t_2]} |x(t)|. \end{aligned} \tag{3.1}$$

Next, choose a real number $\mu > 0$ that satisfies the inequality $(\alpha + \gamma)(1 + 2K)\mu < 1$; note that μ depends only on the specified constants α, γ, and K. Setting

$$t_2 = t_1 + \mu,$$

and moving the last term of (3.1) to the left side of the inequality, we get

$$\begin{aligned} &[1-(\alpha+\gamma)(1+2K)\mu] \sup_{t\in[t_1,t_1+\mu]} |x(t)| \\ &\leq |x(t_1)| + 2(\alpha+\gamma)\mu K \\ &\quad + 2(\alpha+\gamma)\mu K \sup_{t\in[t_1-\tau,t_1]} |x(t)|. \end{aligned}$$

Considering that $|x(t_1)| \leq \sup_{t\in[t_1-\tau,t_1]} |x(t)|$, and defining the ratios

$$\eta_1 := \frac{2(\alpha+\gamma)\mu K}{[1-(\alpha+\gamma)(1+2K)\mu]}, \qquad \eta_2 := \frac{[1+2(\alpha+\gamma)\mu K]}{[1-(\alpha+\gamma)(1+2K)\mu]},$$

yields

$$\sup_{t\in[t_1,t_1+\mu]} |x(t)| \leq \eta_1 + \eta_2 \sup_{t\in[t_1-\tau,t_1]} |x(t)|. \tag{3.2}$$

Note that μ, η_1, and η_2 depend only on the specified constants α, γ, and K.

Next, referring to the activation time $t_a < 0$ of Convention 2.1, let r be the smallest integer satisfying $r \geq (T - t_a)/\mu$; build the partition

$$\begin{aligned} [t_a, T] \subseteq \{&[t_a, t_a+\mu], [t_a+\mu, t_a+2\mu], \ldots, \\ &\times [t_a+(r-1)\mu, t_a+r\mu]\}. \end{aligned} \tag{3.3}$$

Let q be the smallest integer satisfying $q \geq \tau/\mu$, and note that

$$\begin{aligned} \sup_{t\in[t_1-\tau,t_1]} |x(t)| &\leq \sup_{t\in[t_1-q\mu,t_1]} |x(t)| \\ &\leq \sum_{j=0}^{q-1} \left(\sup_{t\in[t_1-(j+1)\mu,t_1-j\mu]} |x(t)| \right). \end{aligned} \tag{3.4}$$

Note also that, by Convention 2.1 and Proposition 2.1, we have

$$x(t) = 0, t \le t_a. \tag{3.5}$$

Furthermore, for an integer k, define the scalar

$$\zeta_k := \sup_{t\in[t_a+(k-1)\mu, t_a+k\mu]} |x(t)|.$$

Then, setting $t_1 := t_a + k\mu$, we obtain from (3.2) and (3.4) the relation

$$\zeta_{k+1} \le \eta_1 + \eta_2 \sum_{j=0}^{q-1} \zeta_{k-j}, \tag{3.6}$$

a linear recursive relation with step counter k. According to (3.5), this relation has the initial conditions

$$\zeta_k = 0 \text{ for all } k \le 0. \tag{3.7}$$

Using well-established properties of linear recursions, it follows by (3.6) and (3.7) that there are real numbers d_1, d_2, … ≥ 0 satisfying

$$|\zeta_k| \le d_k \eta_1, k = 1, 2, \dots, \tag{3.8}$$

where d_1, d_2, … depend only on η_2 and on k. Recalling from (3.3) that $T \le r\mu + t_a$, this shows that

$$\sup_{t\in[0,T]} |x(t)| \le \sup_{t\in[t_a,T]} |x(t)| \le \eta_1 \sum_{k=1}^{r} d_k.$$

Setting $M(T) := \eta_1 \sum_{k=1}^{r} d_k$, we have $\sup_{t\in[0,T]} |x(t)| \le M(T)$. As $M(T)$ depends only on α, γ, K, and T, our proof concludes. ■

Considering that the functions a, b, and c of (2.4) are continuous functions, and recalling that continuous functions map compact domains into bounded domains, we obtain the following implication of Lemma 3.1.

Corollary 3.1: *Let Σ be a system described by (2.4) with the functions a, b, and c, and let $x(t) := \Sigma(x_0, v, u, t)$ be the state of Σ in response to a residual input signal $v \in V_\gamma(v_0)$ and a control input signal $u \in U(K)$. Then, for every time $T \ge 0$, there is a real number $B(T) \ge 0$ such that*

$$\sup_{0\le t\le T} \{|a(t, x(t), x(t-\tau))|, |b(t, x(t), x(t-\tau))|, |c(t, x(t), x(t-\tau))|\} \le B(T)$$

for all $v \in V_\gamma(v_0)$, all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, and all $u \in U(K)$.

3.2 Controllability

Recall that our objective is to bring the controlled system Σ of Figure 1 from its initial state x_0 to a state in the domain $\rho(\ell)$. Whether this is possible or not depends, among other factors, on the relationship between the initial state x_0 and the value of the input bound K. For instance, consider the scalar linear system $\dot{x}(t) = 5x(t) + u(t)$ with the initial state $x_0 = 2$, input amplitude bound $K = 2$, and operating error bound $\ell = 1$. Since the derivative of $x(t)$ cannot be made negative under the given conditions, no input signal with amplitude bounded by 2 can drive the state from $x_0 = 2$ into $\rho(1)$. However, raising the input bound to, say, $K = 11$, makes it possible to bring the state from x_0 into $\rho(1)$. These simple considerations lead us to the following notion.

Definition 3.1: A system Σ with initial state x_0 and residual input v is *K-controllable* if there is a control input signal $u_c \in U(K)$ and a time $t_c \ge 0$ such that $\Sigma(x_0, v, u_c, t_c) = 0$.

The next statement shows that, if the uncertainty parameter γ is not too large, then K-controllability of the nominal system Σ_0 guarantees a somewhat weaker form of controllability for the entire family of systems $\mathcal{F}_\gamma(\Sigma_0)$.

Proposition 3.1: *Assume that the nominal system Σ_0 of (2.9) is K-controllable with the initial state x_0 and the nominal residual input signal v_0. Then, for every real number $\ell > 0$, there is a real number $\gamma > 0$ for which the following is true: there is a control input signal $u_\gamma \in U(K)$ and a time $t_\gamma \ge 0$ such that $\Sigma(x_0, v, u_\gamma, t_\gamma) \in \rho(\ell)$ for all systems $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ and all residual input signals $v \in V_\gamma(v_0)$.*

Proof: As the nominal system Σ_0 is K-controllable with the initial state x_0 and the nominal residual input signal v_0, there are, by Definition 3.1, a control input signal $u_c \in U(K)$ and a time $t_c \ge 0$ such that the state $x(t) := \Sigma_0(x_0, v_0, u_c, t)$ satisfies $x(t_c) = 0$. Using the same control input signal u_c, let $x_\gamma(t) := \Sigma(x_0, v, u_c, t)$ be the response of a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with residual input signal $v \in V_\gamma(v_0)$. Set $\xi(t) := x_\gamma(t) - x(t)$, and note that $\xi(0) = x_\gamma(0) - x(0) = x_0 - x_0 = 0$.

Let t_a be the activation time of Convention 2.1, and consider the input signals

$$w_0(t) := \begin{cases} 0 & \text{for } t \le t_a, \\ v_0(t) & \text{for } t_a \le t \le 0, \\ u_c(t) & \text{for } t \ge 0; \end{cases}$$

$$w_\gamma(t) := \begin{cases} 0 & \text{for } t \le t_a, \\ v(t) & \text{for } t_a \le t \le 0, \\ u_c(t) & \text{for } t \ge 0. \end{cases} \tag{3.9}$$

Note that $w_0(t)$ and $w_\gamma(t)$ differ only in the residual input. Let t_1 and t_2 be two times, where $t_a \le t_1 < t_2 \le t_c$, and let $t \in (t_1, t_2]$. Then, using (2.4), (2.6), and (2.9), and recalling from (2.7) and (2.8) that $a_0(s,0,0) = a_\gamma(s,0,0) = 0$, we can write

$$\begin{aligned}\xi(t) = \xi(t_1) &+ \int_{t_1}^{t} [a_0(s, x_\gamma(s), x_\gamma(s-\tau)) \\ &- a_0(s, x(s), x(s-\tau))]ds \\ &+ \int_{t_1}^{t} [a_\gamma(s, x_\gamma(s), x_\gamma(s-\tau)) - a_\gamma(s,0,0)]ds \\ &+ \int_{t_1}^{t} [b_0(s, x_\gamma(s), x_\gamma(s-\tau)) \\ &- b_0(s, x(s), x(s-\tau))]w_\gamma(s)ds \\ &+ \int_{t_1}^{t} [b_\gamma(s, x_\gamma(s), x_\gamma(s-\tau)) \\ &- b_\gamma(s,0,0)]w_\gamma(s)ds + \int_{t_1}^{t} b_\gamma(s,0,0)w_\gamma(s)ds \\ &+ \int_{t_1}^{t} [b_0(s, x(s), x(s-\tau)) - b_0(s,0,0)][w_\gamma(s) \\ &- w_0(s)]ds + \int_{t_1}^{t} b_0(s,0,0)[w_\gamma(s) - w_0(s)]ds \\ &+ \int_{t_1}^{t} [c_0(s, x_\gamma(s), x_\gamma(s-\tau)) \\ &- c_0(s, x(s), x(s-\tau))]w_\gamma(s-\tau)ds \\ &+ \int_{t_1}^{t} [c_\gamma(s, x_\gamma(s), x_\gamma(s-\tau)) \\ &- c_\gamma(s,0,0)]w_\gamma(s-\tau)ds \\ &+ \int_{t_1}^{t} c_\gamma(s,0,0)w_\gamma(s-\tau)ds \\ &+ \int_{t_1}^{t} [c_0(s, x(s), x(s-\tau)) \\ &- c_0(s,0,0)][w_\gamma(s-\tau) - w_0(s-\tau)]ds \\ &+ \int_{t_1}^{t} c_0(s,0,0)[w_\gamma(s-\tau) - w_0(s-\tau)]ds.\end{aligned}$$

Now, apply (2.7) and (2.8); recall that $|w_\gamma(s)| \le K$ and $|w_0(s)| \le K$; note that $|w_\gamma(s) - w_0(s)| \le \gamma$ for all s by (3.9) and (2.10); and let t vary between t_1 and t_2. This yields

$$\begin{aligned}\sup_{t\in[t_1,t_2]} |\xi(t)| \le |\xi(t_1)| &+ \alpha \sup_{t\in[t_1-\tau,t_2]} |\xi(t)|(t_2 - t_1) \\ &+ \gamma \sup_{t\in[t_1-\tau,t_2]} |x_\gamma(t)|(t_2 - t_1) \\ &+ \alpha \sup_{t\in[t_1-\tau,t_2]} |\xi(t)|K(t_2 - t_1) \\ &+ \gamma \sup_{t\in[t_1-\tau,t_2]} |x_\gamma(t)|K(t_2 - t_1) + \gamma K(t_2 - t_1) \\ &+ \alpha \sup_{t\in[t_1-\tau,t_2]} |x(t)|\gamma(t_2 - t_1) \\ &+ \alpha\gamma(t_2 - t_1) + \alpha \sup_{t\in[t_1-\tau,t_2]} |\xi(t)|K(t_2 - t_1) \\ &+ \gamma \sup_{t\in[t_1-\tau,t_2]} |x_\gamma(t)|K(t_2 - t_1) + \gamma K(t_2 - t_1) \\ &+ \alpha \sup_{t\in[t_1-\tau,t_2]} |x(t)|\gamma(t_2 - t_1) + \alpha\gamma(t_2 - t_1).\end{aligned} \tag{3.10}$$

Next, for any function $g(t)$, we can write

$$\sup_{t\in[t_1-\tau,t_2]} |g(t)| \le \sup_{t\in[t_1-\tau,t_1]} |g(t)| + \sup_{t\in[t_1,t_2]} |g(t)| \tag{3.11}$$

Applying (3.11) to the terms in (3.10), denoting $\mu := t_2 - t_1$, and using the bound $M(t_c)$ of Lemma 3.1, we obtain

$$\begin{aligned}\sup_{t\in[t_1,t_2]} |\xi(t)| \le |\xi(t_1)| &+ \alpha(1+2K)\mu \\ &\times \left[\sup_{t\in[t_1,t_2]} |\xi(t)| + \sup_{t\in[t_1-\tau,t_1]} |\xi(t)|\right] \\ &+ \gamma\,[(1+2K+2\alpha)M(t_c) + 2(K+\alpha)]\,\mu.\end{aligned} \tag{3.12}$$

Next, choose the number μ to satisfy $\alpha(1+2K)\mu < 1$, and define the numbers

$$\eta_1 := \frac{1}{1-\alpha(1+2K)\mu},\ \eta_2 := \frac{\alpha(1+2K)\mu}{1-\alpha(1+2K)\mu},$$
$$\eta_3 := \frac{[(1+2K+2\alpha)M(t_c) + 2(K+\alpha)]\mu}{1-\alpha(1+2K)\mu}.$$

Note that μ, η_1, η_2, and η_3 are positive and depend only on the specified parameters α, γ, and K. Substituting into (3.12), we get

$$\sup_{t\in[t_1,t_1+\mu]} |\xi(t)| \le \eta_1|\xi(t_1)| + \eta_2 \sup_{t\in[t_1-\tau,t_1]} |\xi(t)| + \gamma\eta_3 \tag{3.13}$$

Furthermore, let $r \ge (t_c - t_a)/\mu$ and $q \ge \tau/\mu$ be the smallest integers satisfying these inequalities. Build the partition

$$[t_a - \tau, t_c] \subseteq \{[t_a - q\mu, t_a - (q-1)\mu], [t_a - (q-1)\mu, t_a - (q-2)\mu], \dots, [t_a + (r-1)\mu, t_a + r\mu]\}.$$

For an integer j, define the real number

$$\zeta_j := \sup_{t\in[t_a+(j-1)\mu,\, t_a+j\mu]} |\xi(t)|. \tag{3.14}$$

Now, select $t_1 := t_a + k\mu$ and note that, for this t_1, we have $|\xi(t_1)| \le \zeta_k$. Then, we can rewrite (3.13) in the form

$$\zeta_{k+1} \le \eta_1\zeta_k + \eta_2\sum_{j=0}^{q-1}\zeta_{k-j} + \gamma\eta_3,\ k = -q, -q+1, \dots, r-1,$$
$$\zeta_k = 0 \text{ for all } k \le -q.$$

Using well-established features of linear recursions and an argument similar to the one used in the proof of Lemma 3.1, we conclude that there is a real number $M \geq 0$ such that $\zeta_k \leq \gamma M$ for all $k \in \{-q, -q+1, \ldots, r\}$. In view of (3.14), this yields

$$\sup_{t\in[t_a,t_c]} |\xi(t)| \leq \gamma M. \tag{3.15}$$

In particular, $|\xi(t_c)| \leq \gamma M$. Thus, referring to (2.11) and recalling that the state of Σ is of dimension n, it follows that every $\gamma > 0$ satisfying

$$\gamma < \frac{\sqrt{\ell/n}}{M}$$

fulfils the statement of the proposition at the time $t_\gamma = t_c$ with the control input signal $u_\gamma = u_c$. This concludes our proof. ∎

Proposition 3.1 shows that, as long as uncertainties are not excessive, K-controllability of the nominal system Σ_0 guarantees that there is a control input signal that brings all members of the family $\mathcal{F}_\gamma(\Sigma_0)$ into the domain $\rho(\ell)$ at the same time. For such uncertainty, we show in the next section that an optimal control input signal $u^*(x_0, \gamma, \ell)$ exists.

4. Optimal solutions

We show now that an optimal solution of Problem 2.1*(i)* exists under rather general conditions, as follows.

Theorem 4.1: *Assume that the nominal system Σ_0 of (2.9) is K-controllable with the initial state x_0 and the nominal residual input signal v_0, and let $\gamma > 0$ be a real number satisfying the condition of Proposition 3.1 for the operating error bound $\ell > 0$. Then,*

(i) There is a finite minimal time $t^*(x_0, \gamma, \ell)$, and
(ii) There is an optimal control input signal $u^*(x_0, \gamma, \ell) \in U(K)$ satisfying $t^*(x_0, \gamma, \ell) = t(x_0, \gamma, \ell, u^*(x_0, \gamma, \ell))$.

The proof of Theorem 4.1 is stated later in this section. It is based on the Generalised Weierstrass Theorem, which, crudely stated, says that a continuous functional attains extremal values in a compact domain. In our case, it turns out that the set of input signals $U(K)$ is compact in a certain sense and the minimal time $t(x_0, \gamma, \ell, u)$ of (2.13) is a continuous functional of the control input signal u (in a certain sense). The Generalised Weierstrass Theorem then implies that the functional $t(x_0, \gamma, \ell, u)$ attains a minimum within $U(K)$. This proves the existence of an optimal control input signal $u^*(x_0, \gamma, \ell)$. To formalise these arguments, we need to review a few notions (e.g. Lusternik & Sobolev, 1961).

Definition 4.1: Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

(i) A sequence $u_1, u_2, \ldots$ of members of H *converges weakly* to a member $u \in H$ if $\lim_{i\to\infty}\langle u_i, y\rangle = \langle u, y\rangle$ for every $y \in H$.
(ii) A subset W of H is *weakly compact* if every sequence of members of W has a subsequence that converges weakly to a member of W.

The following statement is from Chakraborty and Hammer (2009b, 2010).

Lemma 4.1: *The set of control input signals $U(K)$ of (2.2) is weakly compact in the topology of the Hilbert space $L_2^{\omega,m}$.*

Our discussion depends on the following weaker notions of continuity (e.g. Zeidler, 1985).

Definition 4.2: Let S be a subset of a Hilbert space H, and let z be a point of S. A functional F: $S \to R$ is *weakly lower semi-continuous* at z if the following holds true whenever F is finite at z: for every sequence $\{z_i\}_{i=1}^\infty \subseteq S$ that converges weakly to z and for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $F(z) - F(z_i) < \varepsilon$ for all $i \geq N$. If F is weakly lower semi-continuous at every point $z \in S$, then F is weakly lower semi-continuous over S.

A function G: $S \times R \to R^n : (s, t) \mapsto G(s, t)$ is *weakly continuous* at a point $z \in S$ at a time t if the following is true for every sequence $\{z_i\}_{i=1}^\infty \subseteq S$ that converges weakly to z: for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z, t) - G(z_i, t)| < \varepsilon$ for all $i \geq N$. If G is weakly continuous at every point $z \in S$ at the time t, then G is weakly continuous over S at the time t.

The function G is *uniformly weakly continuous* over a time interval $[t_1, t_2]$ if the following is true for every point $z \in S$ and for every sequence $\{z_i\}_{i=1}^\infty \subseteq S$ that converges weakly to z: for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z, t) - G(z_i, t)| < \varepsilon$ for all $i \geq N$ at all times $t \in [t_1, t_2]$.

The following statement shows that all systems under consideration are weakly continuous functions of the control input signal.

Lemma 4.2: *Let Σ be a member of the family $\mathcal{F}_\gamma(\Sigma_0)$, and let $v \in V_\gamma(v_0)$ be a residual input signal. Then, the response $\Sigma(x_0, v, u, t) : U(K) \to R^n : u \mapsto \Sigma(x_0, v, u, t)$ forms a uniformly weakly continuous function over every finite interval of time.*

Proof: Let $\{u_i\}_{i=1}^\infty \subseteq U(K)$ be a sequence of control input signals that converges weakly to a control input signal $u \in U(K)$, and fix a time $t \geq 0$. We show first that the sequence $\{\Sigma(x_0, v, u_i, t)\}_{i=1}^{\infty}$ converges to $\Sigma(x_0, v, u, t)$. Referring to the activation time t_a of Convention 2.1, define, for each integer $i = 1, 2, \ldots$, the signals

$$w(t) = \begin{cases} v(t) & t \in [t_a, 0), \\ u(t) & t \geq 0; \end{cases}$$

$$w_i(t) = \begin{cases} v(t) & t \in [t_a, 0), \\ u_i(t) & t \geq 0. \end{cases}$$

Denote $x(t, w) := \Sigma(x_0, v, u, t)$ and $x(t, w_i) := \Sigma(x_0, v, u_i, t)$, and consider the difference $x(t, i) := x(t, w) - x(t, w_i)$, $i = 1, 2, \ldots$. As $x(t, w_i)$ and $x(t, w)$ start from the initial state $x(0) = x_0$, we have $x(0, i) = 0$, $i = 1, 2, \ldots$. Now, let $t_2 > t_1 \geq 0$ be two times, and let $t \in (t_1, t_2]$. Then, using (2.4), we can write

$$\begin{aligned} x(t, i) = x(t_1, i) &+ \int_{t_1}^{t} [a(s, x(s, w), x(s-\tau, w)) \\ &- a(s, x(s, w_i), x(s-\tau, w_i))]ds \\ &+ \int_{t_1}^{t} [b(s, x(s, w)), x(s-\tau, w)) \\ &- b(s, s, x(s, w_i), x(s-\tau, w_i))]w_i(s)ds \\ &+ \int_{t_1}^{t} b(s, x(s, w), x(s-\tau, w))(w(s) \\ &- w_i(s))ds + \int_{t_1}^{t} [c(s, x(s, w), x(s-\tau, w)) \\ &- c(s, x(s, w_i), x(s-\tau, w_i))]w_i(s-\tau)ds \\ &+ \int_{t_1}^{t} c(s, x(s, w), x(s-\tau, w))(w(s-\tau) \\ &- w_i(s-\tau))ds. \end{aligned}$$

Applying (2.7) and (2.8) together with the fact that $|w(t)| \leq K$ and $|w_i(t)| \leq K$ for all t, we get

$$\begin{aligned} \sup_{t_1 \le t \le t_2} |x(t, i)| &\leq |x(t_1, i)| + (\alpha + \gamma) \sup_{t_1-\tau \le t \le t_2} |x(t, i)|(t_2 - t_1) \\ &+ 2(\alpha + \gamma) \sup_{t_1-\tau \le t \le t_2} |x(t, i)|K(t_2 - t_1) \\ &+ \sup_{t_1 \le t \le t_2} \left| \int_{t_1}^{t} b(s, x(s, w), x(s-\tau, w))(w(s) - w_i(s))ds \right| \\ &+ \sup_{t_1 \le t \le t_2} \left| \int_{t_1}^{t} c(s, x(s, w), x(s-\tau, w))(w(s-\tau) \right. \\ &\left. - w_i(s-\tau))ds \right|. \end{aligned} \quad (4.1)$$

Using the fact that $\sup_{t_1-\tau \le t \le t_2} |x(t, i)| \leq \sup_{t_1-\tau \le t \le t_1} |x(t, i)| + \sup_{t_1 \le t \le t_2} |x(t, i)|$, and rearranging terms, we obtain

$$\begin{aligned} &(1 - (\alpha + \gamma)(1 + 2K)(t_2 - t_1)) \sup_{t_1 \le t \le t_2} |x(t, i)| \\ &\leq |x(t_1, i)| + (\alpha + \gamma)(1 + 2K)(t_2 - t_1) \sup_{t_1-\tau \le t \le t_1} |x(t, i)| \\ &\quad + \sup_{t_1 \le t \le t_2} \left| \int_{t_1}^{t} b(s, x(s, w), x(s-\tau, w))(w(s) - w_i(s))ds \right| \\ &\quad + \sup_{t_1 \le t \le t_2} \left| \int_{t_1}^{t} c(s, x(s, w), x(s-\tau, w))(w(s-\tau) \right. \\ &\quad \left. - w_i(s-\tau))ds \right|. \end{aligned} \quad (4.2)$$

Now, let $\mu > 0$ be a real number satisfying $(1 - (\alpha + \gamma)(1 + 2K)\mu) > 0$, and set $t_2 := t_1 + \mu$. Denote

$$\begin{aligned} \eta_4 &:= (1 - (\alpha + \gamma)(1 + 2K)\mu)^{-1}, \\ \eta_5 &:= \eta_4(\alpha + \gamma)(1 + 2K)\mu, \end{aligned} \quad (4.3)$$

and note that $|x(t_1, i)| \leq \sup_{t_1-\tau \le t \le t_1} |x(t, i)|$. Then, (4.2) can be rewritten in the form

$$\begin{aligned} \sup_{t_1 \le t \le t_2} |x(t, i)| &\leq (\eta_4 + \eta_5) \sup_{t_1-\tau \le t \le t_1} |x(t, i)| \\ &+ \eta_4 \sup_{t_1 \le t \le t_2} \left| \int_{t_1}^{t} b(s, x(s, w), x(s-\tau, w))(w(s) \right. \\ &\left. - w_i(s))ds \right| \\ &+ \eta_4 \sup_{t_1 \le t \le t_2} \left| \int_{t_1}^{t} c(s, x(s, w), x(s-\tau, w))(w(s-\tau) \right. \\ &\left. - w_i(s-\tau))ds \right|. \end{aligned} \quad (4.4)$$

To examine the last two terms of (4.4), define two matrix functions $y_t^b, y_t^c : R \to R^{m \times n}$ at a fixed time $t \geq t_1$:

$$y_t^b(s) := \begin{cases} e^{\omega s} b^T(s, x(s, w), x(s-\tau, w)) & t_1 \leq s \leq t, \\ 0 & \text{else.} \end{cases}$$

$$y_t^c(s) := \begin{cases} e^{\omega s} c^T(s, x(s, w), x(s-\tau, w)) & t_1 \leq s \leq t, \\ 0 & \text{else.} \end{cases}$$

Then, denoting $w_\tau(s) := w(s - \tau)$ and $w_{\tau i}(s) := w_i(s - \tau)$, and using the inner product (2.1), (2.3), we can write

$$\begin{aligned} &\sup_{t_1 \le t \le t_2} \left| \int_{t_1}^{t} b(s, x(s, w), x(s-\tau, w))[w(s) - w_i(s)]ds \right| \\ &= \sup_{t_1 \le t \le t_2} \left| \langle y_t^b, w - w_i \rangle \right|, \\ &\sup_{t_1 \le t \le t_2} \left| \int_{t_1}^{t} c(s, x(s, w), x(s-\tau, w))[w(s-\tau) \right. \\ &\left. - w_i(s-\tau)]ds \right| = \sup_{t_1 \le t \le t_2} \left| \langle y_t^c, w_\tau - w_{\tau i} \rangle \right|. \end{aligned}$$

Now, as the sequence $\{w_i\}_{i=1}^{\infty}$ converges weakly to w, it is also true that the sequence $\{w_{\tau i}\}_{i=1}^{\infty}$ converges weakly to w_τ. Therefore, there is, for every real number $\beta > 0$, an integer $N_t > 0$ for which $\max\left\{\left|\langle y_t^b, w - w_i\rangle\right|, \left|\langle y_t^c, w_\tau - w_{\tau i}\rangle\right|\right\} < \beta$ for all $i \geq N_t$. We must show that N_t can be chosen independently of t, i.e. that there is an integer $N > 0$ such that

$$\sup_{t_1 \le t \le t_1+\mu} \left|\langle y_t^b, w - w_i\rangle\right| < \beta \text{ and } \sup_{t_1 \le t \le t_1+\mu} \left|\langle y_t^c, w_\tau - w_{\tau i}\rangle\right| < \beta \text{ for all } i \geq N. \tag{4.5}$$

To prove the latter, we start with the first term of (4.5). Assume, by contradiction, that there is a sequence of times $\{\theta_j\}_{j=1}^{\infty} \subseteq [t_1, t_1 + \mu]$ for which

$$\left|\left\langle y_{\theta_j}^b, w - w_j\right\rangle\right| \geq \beta \text{ for all } j = 1, 2, \ldots \tag{4.6}$$

As the interval $[t_1, t_1 + \mu]$ is compact, the sequence $\{\theta_j\}_{j=1}^{\infty}$ includes a convergent subsequence, say, $\{\theta_{j_k}\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} \theta_{j_k} = \theta' \in [t_1, t_1 + \mu]$. In addition, since the sequence $\{w_i\}_{i=1}^{\infty}$ converges weakly to w, so does its subsequence $\{w_{j_k}\}_{k=1}^{\infty}$. Consequently, there is an integer $N' > 0$ such that $\left|\langle y_{\theta'}^b, w - w_{j_k}\rangle\right| < \beta/2$ for all $k \geq N'$. Employing the bound $B(t_1 + \mu)$ of Corollary 3.1 together with (2.7) and (2.8), and recalling that all input signals are bounded by K, we can write

$$\begin{aligned}
&\left|\left\langle y_{\theta'}^b, w - w_{j_k}\right\rangle - \left\langle y_{\theta_{j_k}}^b, w - w_{j_k}\right\rangle\right| \\
&= \left|\int_{\theta_{j_k}}^{\theta'} b(s, x(s, w), x(s - \tau, w))[w(s) - w_{j_k}(s)]ds\right| \\
&\leq B(t_1 + \mu)K\left|\theta' - \theta_{j_k}\right|.
\end{aligned}$$

Now, using the fact that $\lim_{k\to\infty} \theta_{j_k} = \theta'$, select an integer $N^* \geq N'$ satisfying $|\theta' - \theta_{j_k}| < \beta/[2B(t_1 + \mu)K]$ for all $k \geq N^*$. Then,

$$\begin{aligned}
\left|\left\langle y_{\theta_{j_k}}^b, w - w_{j_k}\right\rangle\right| &= \left|\left\langle y_{\theta_{j_k}}^b, w - w_{j_k}\right\rangle - \left\langle y_{\theta'}^b, w - w_{j_k}\right\rangle + \left\langle y_{\theta'}^b, w - w_{j_k}\right\rangle\right| \\
&\leq \left|\left\langle y_{\theta_{j_k}}^b, w - w_{j_k}\right\rangle - \left\langle y_{\theta'}^b, w - w_{j_k}\right\rangle\right| + \left|\left\langle y_{\theta'}^b, w - w_{j_k}\right\rangle\right| \\
&< \beta/2 + \beta/2 = \beta
\end{aligned}$$

for all $k \geq N^*$, contradicting (4.6). Consequently, for every real number $\beta > 0$, there is an integer $N_b > 0$ such that $\sup_{t_1 \le \theta \le t_1+\mu} \left|\langle y_\theta^b, w - w_i\rangle\right| < \beta$ for all $i \geq N_b$.

Applying a similar argument to the term with the function c, we conclude that, for every real number $\beta > 0$, there is an integer $N_c > 0$ such that $\sup_{t_1 \le \theta \le t_1+\mu} \left|\langle y_\theta^c, w_\tau - w_{\tau i}\rangle\right| < \beta$ for all $i \geq N_c$. Thus, setting $N := \max\{N_b, N_c\}$, we obtain that, for every real number $\beta > 0$, there is an integer $N > 0$ such that

$$\sup_{t_1 \le t \le t_1+\mu} \max\left\{\left|\langle y_t^b, w - w_i\rangle\right|, \left|\langle y_t^c, w_\tau - w_{\tau i}\rangle\right|\right\} < \beta \text{ for all } i \geq N. \tag{4.7}$$

To continue, fix a real number $\varepsilon > 0$ and, recalling the number η_4 of (4.3), select a real number β satisfying $0 < \beta < \varepsilon/(2\eta_4)$. By (4.4) and (4.7), this leads to the conclusion that, for every real number $\varepsilon > 0$, there is an integer $N_\varepsilon \geq 0$ such that

$$\sup_{t_1 \le \theta \le t_1+\mu} |x(\theta, i)| \leq \varepsilon + (\eta_4 + \eta_5) \times \sup_{t_1-\tau \le \theta \le t_1} |x(\theta, i)| \text{ for all } i \geq N_\varepsilon. \tag{4.8}$$

Invoking the similarity between (4.8) and (3.2), we can use an argument similar to the one following (3.2) to define the sequence

$$\zeta_k := \sup_{\theta \in [t_a+(k-1)\mu, t_a+k\mu]} |x(\theta, i)|, \; k = \ldots, -1, 0, 1, \ldots,$$

which satisfies the linear recursion

$$\zeta_{k+1} \leq \varepsilon + (\eta_4 + \eta_5)\sum_{j=0}^{q-1} \zeta_{k-j}, \; \zeta_k = 0 \text{ for all } k \leq 0.$$

By properties of linear recursions, we conclude that there is an expression $H(\eta_4 + \eta_5, k)$ consisting of a combination of the powers $(\eta_4 + \eta_5), (\eta_4 + \eta_5)^2, \ldots, (\eta_4 + \eta_5)^k$, for which $\zeta_k \leq |H(\eta_4 + \eta_5, k)|\varepsilon$. Thus, by choosing ε sufficiently small, we can make ζ_k as small as desired, and it follows that

$$\lim_{i\to\infty}\left(\sup_{0 \le \theta \le t} |x(\theta, i)|\right) = 0 \tag{4.9}$$

for every finite time $t \geq 0$, so that

$$\lim_{i\to\infty} x(\theta, w_i) = x(\theta, w)$$

for all times $\theta \in [0, t]$. Furthermore, by (4.9), this convergence is uniform over the interval $[t_1, t_2]$. This concludes our proof. ■

To continue, we review a few more facts (e.g. Willard, 2004).

Theorem 4.2:

(*i*) *A weakly continuous functional is also weakly lower semi-continuous.*

(*ii*) *Let S and A be topological spaces and assume that, for every member $a \in A$, there is a weakly lower semi-continuous functional f_a: $S \to R$. If $\sup_{a \in A} f_a(s)$ exists at every point $s \in S$, then the functional $f(s) := \sup_{a \in A} f_a(s)$ is weakly lower semi-continuous on S.*

Fix a time $t \geq 0$ and consider the functional $\psi(t, \cdot)$: $U(K) \to R$: $u \mapsto \psi(t, u)$ given by

$$\psi(t, u) := \sup_{(\Sigma, v) \in \mathcal{F}_\gamma(\Sigma_0) \times V_\gamma(v_0)} \Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t). \tag{4.10}$$

Now, by Lemma 4.2, the function $\Sigma(x_0, v, u, t)$: $U(K) \to R^n$ is weakly continuous. Since a continuous function of a weakly continuous function is weakly continuous, it follows that the functional $\Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t)$: $U(K) \to R$ is weakly continuous. By Theorem 4.2*(i)*, this functional is also weakly lower semi-continuous. Thus, $\psi(t, u)$ is weakly lower semi-continuous by Theorem 4.2*(ii)*; this verifies the following statement.

Lemma 4.3: *The functional $\psi(t, u)$ of (4.10) is weakly lower semi-continuous over $U(K)$ at every time $t \geq 0$.*

According to (2.13), the minimal time $t(x_0, \gamma, \ell, u)$ can be expressed in the form

$$t(x_0, \gamma, \ell, u) = \inf_{t \geq 0} \{\psi(t, u) \leq \ell\}.$$

Based on this expression and Lemma 4.3, a proof identical to the one used in Yu and Hammer (2016a, proof of Proposition 3.4) establishes the following statement.

Proposition 4.1: *The functional $t(x_0, \gamma, \ell, u)$: $U(K) \to R$ is weakly lower semi-continuous over $U(K)$ at every time $t \geq 0$.*

To continue, we need the following version of the Generalised Weierstrass Theorem (e.g. Willard, 2004).

Theorem 4.3: *A weakly lower semi-continuous functional attains its minimum in a weakly compact set.*

Proof of Theorem 4.1: By Lemma (4.1), the set $U(K)$ is weakly compact, and, by Proposition 4.1, the time functional $t(x_0, \gamma, \ell, u)$ is weakly lower semi-continuous over $U(K)$. Thus, it follows from Theorem 4.3 that the minimal time $t^*(x_0, \gamma, \ell)$ of (2.14) is achieved by a control input signal $u^*(x_0, \gamma, \ell) \in U(K)$. This completes the proof. ■

Theorem 4.1 shows that Problem 2.1*(i)* has an optimal solution $u^*(x_0, \gamma, \ell)$ under rather general conditions; the only requirements are that the nominal system Σ_0 be K-controllable and that the uncertainty parameter γ not be too big.

Still, an optimal control input signal $u^*(x_0, \gamma, \ell)$, is, in general, a vector-valued function of time; it may be hard to calculate and implement. The next section shows that $u^*(x_0, \gamma, \ell)$ can be replaced by a bang-bang signal, without significantly degrading performance.

5. Practical implementation

This section addresses part *(ii)* of Problem 2.1. We show that an optimal control input signal $u^*(x_0, \gamma, \ell)$ can be replaced by a bang-bang signal $u^\pm$, without a significant departure from optimal performance. Once the existence of such bang-bang signals is assured, they can be derived through relatively simple numerical algorithms.

To introduce the main statement of this section, let $\ell' > \ell$ be an operating error bound that is slightly larger than the specified operating error bound ℓ. We show that there is a bang-bang control input signal $u^\pm \in U(K)$ which, for the error bound ℓ', achieves a time $t(x_0, \gamma, \ell', u^\pm)$ that is no longer than the optimal time $t^*(x_0, \gamma, \ell)$ for the original error bound ℓ. Thus, bang-bang control input signals can achieve performance that is as close as desired to optimal performance.

Theorem 5.1: *Assume that the nominal system Σ_0 is K-controllable from the initial state x_0 with the nominal residual input signal v_0, and let $\ell > 0$ be the specified operating error bound. Then, for every $\ell' > \ell$, there are $\gamma > 0$ and bang-bang control input signal $u^\pm \in U(K)$ (with a finite number of switchings) such that $t(x_0, \gamma, \ell', u^\pm) \leq t^*(x_0, \gamma, \ell)$.*

The proof of Theorem 5.1 depends on the next statement, which states that the response to any control input signal can be approximated by the response to a bang-bang control input signal.

Theorem 5.2: *Let Σ be a member of $\mathcal{F}_\gamma(\Sigma_0)$, and let $u \in U(K)$ be a control input signal of Σ. Let $t' \geq 0$ be a time, and let $\sigma > 0$ be a real number. Then, there are a real number $\gamma > 0$ and a bang-bang control input signal $u^\pm \in U(K)$ (with a finite number of switchings) such that the following is true: the difference between the response $x(t) := \Sigma(x_0, v, u, t)$ of Σ to u and the response $x^\pm(t) := \Sigma(x_0, v, u^\pm, t)$ of Σ to $u^\pm$ satisfies $|x(t) - x^\pm(t)| < \sigma$ for all $t \in [0, t']$, for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, and for all $v \in V_\gamma(v_0)$.*

The proof of Theorem 5.2 requires the following.

Lemma 5.1: *Let Σ be a system of the form (2.4) with the functions b and c. Denote $x(t) := \Sigma(x_0, v, u, t)$, and fix a time $t' > 0$. Then, for every real number $\varepsilon > 0$, there are real numbers $\beta(t', \varepsilon) > 0$ and $\gamma > 0$ such that the following* *are true for all residual input signals $v \in V(v_0, \gamma)$, for all control input signals $u \in U(K)$, and for all systems $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$:*

$$|b(t_1, x(t_1), x(t_1 - \tau)) - b(t_2, x(t_2), x(t_2 - \tau))| < \varepsilon \text{ and}$$
$$|c(t_1, x(t_1), x(t_1 - \tau)) - c(t_2, x(t_2), x(t_2 - \tau))| < \varepsilon, \quad (5.3)$$

where $t_1, t_2 \in [0, t']$ are any times satisfying $|t_1 - t_2| < \beta(t', \varepsilon)$.

Proof: Let $v \in V(v_0, \gamma)$ be a residual input signal, let $u \in U(K)$ be a control input signal, let w be the combined input signal of (2.5), and refer to the functions b and c of (2.4). As $x(t) = \Sigma(x_0, w, t)$ originates from the integration of bounded Lebesgue measurable functions, it is a continuous function of t. As b and c are continuous functions as well, it follows that $b(t, x(t), x(t - \tau))$ and $c(t, x(t), x(t - \tau))$ are continuous functions of t, and hence are uniformly continuous over the compact interval $[t_a, t']$, where t_a is the activation time of Convention 2.1. Consequently, for every real number $\varepsilon > 0$, there is a real number $\beta(\varepsilon, w) > 0$ for which $|b(t_1, x(t_1), x(t_1 - \tau)) - b(t_2, x(t_2), x(t_2 - \tau))| < \varepsilon$ and $|c(t_1, x(t_1), x(t_1 - \tau)) - c(t_2, x(t_2), x(t_2 - \tau))| < \varepsilon$ for every pair of times $t_1, t_2 \in [t_a, t']$ satisfying $|t_1 - t_2| < \beta(\varepsilon, w)$. We show that $\beta(\cdot, w)$ can be selected independently of w and Σ.

To this end, let $\varepsilon' \in (0, \varepsilon)$ be a number and denote

$$\beta'(\varepsilon', w) := \sup\left\{ |t_1 - t_2| \;\middle|\; \begin{array}{l} t_1, t_2 \in [t_a, t'], \\ |b(t_1, x(t_1), x(t_1 - \tau)) - b(t_2, x(t_2), x(t_2 - \tau))| < \varepsilon', \\ |c(t_1, x(t_1), x(t_1 - \tau)) - c(t_2, x(t_2), x(t_2 - \tau))| < \varepsilon'. \end{array} \right\}$$

Consider the infimum of $\beta'(\varepsilon', w)$ over all permissible input signals:

$$\beta^*(\varepsilon') := \inf_{w : w(t - t_a) \in U(K)} \beta'(\varepsilon', w). \quad (5.4)$$

If $\beta^*(\varepsilon') > 0$ for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, then the lemma is valid for $\beta(t', \varepsilon) = \beta^*(\varepsilon')$, and our proof concludes. To prove that this is the case, we show first that $\beta^*(\varepsilon') = 0$ is not a valid option.

By contradiction, assume that $\beta^*(\varepsilon') = 0$. Then, there is a sequence of signals $\{w_i\}_{i=1}^\infty$, where $w_i(t - t_a) \in U(K)$, $i = 1, 2, \ldots$, such that $\lim_{i \to \infty} \beta(\varepsilon', w_i) = 0$. By Lemma 4.1, the sequence of signals $z_i(t) := w_i(t - t_a)$, $i = 1, 2, \ldots$, has a weakly convergent subsequence $\{z_{i_k}\}_{k=1}^\infty$ that converges weakly to a signal $z \in U(K)$. Set $w(t) := z(t + t_a)$ and apply Lemma 4.2 (see, in particular, (4.9)); it follows that, for every real number $\delta > 0$, there is an integer $N_\delta > 0$ such that

$$\sup_{t \in [0, t']} \{|\Sigma(x_0, w_{i_k}, t) - \Sigma(x_0, w, t)|, |\Sigma(x_0, w_{i_k}, t - \tau) - \Sigma(x_0, w, t - \tau)|\} < \delta \quad (5.5)$$

for all $k \geq N_\delta$. Now, by Lemma 3.1, we have that $|x(t)| \leq M(t')$ at all times $t \leq t'$. Invoking this bound, it follows by uniform continuity of the functions $b(t, y, z)$ and $c(t, y, z)$ over the compact domain $[t_a, t'] \times [-M(t'), M(t')]^n \times [-M(t'), M(t')]^n$, that, for every real number $\delta' > 0$, there is a real number $\varepsilon'' > 0$ such that $|b(t, y, z) - b(t, y', z')| < \delta'/4$ and $|c(t, y, z) - c(t, y', z')| < \delta'/4$ whenever $|y - y'| < \varepsilon''$, $|z - z'| < \varepsilon''$, and $|y|, |y'|, |z|, |z'| \leq M(t')$. Selecting $\delta = \varepsilon''$ in (5.5), it follows that, for all integers $k \geq N_{\varepsilon''}$, we have

$$|b(t, \Sigma(x_0, w_{i_k}, t), \Sigma(x_0, w_{i_k}, t - \tau)) - b(t, \Sigma(x_0, w, t), \Sigma(x_0, w, t - \tau))| < \delta'/4 \text{ and}$$
$$|c(t, \Sigma(x_0, w_{i_k}, t), \Sigma(x_0, w_{i_k}, t - \tau)) - c(t, \Sigma(x_0, w, t), \Sigma(x_0, w, t - \tau))| < \delta'/4 \quad (5.6)$$

for all $t \in [t_a, t']$.

Furthermore, by uniform continuity of the functions $b(t, x(t), x(t - \tau))$ and $c(t, x(t), x(t - \tau))$ over the compact time interval $[t_a, t']$, there is a real number $\beta > 0$ such that

$$|b(t_1, \Sigma(x_0, w, t_1), \Sigma(x_0, w, t_1 - \tau)) - b(t_2, \Sigma(x_0, w, t_2), \Sigma(x_0, w, t_2 - \tau))| < \delta'/4 \text{ and}$$
$$|c(t_1, \Sigma(x_0, w, t_1), \Sigma(x_0, w, t_1 - \tau)) - c(t_2, \Sigma(x_0, w, t_2), \Sigma(x_0, w, t_2 - \tau))| < \delta'/4$$

for all $t_1, t_2 \in [t_a, t']$ satisfying $|t_1 - t_2| < \beta$. Combining this with (5.6), we obtain

$$\begin{aligned} &\left|b(t_1, \Sigma(x_0, w_{i_k}, t_1), \Sigma(x_0, w_{i_k}, t_1 - \tau)) - b(t_2, \Sigma(x_0, w_{i_k}, t_2), \Sigma(x_0, w_{i_k}, t_2 - \tau))\right| \\ &\leq \left|b(t_1, \Sigma(x_0, w_{i_k}, t_1), \Sigma(x_0, w_{i_k}, t_1 - \tau)) - b(t_1, \Sigma(x_0, w, t_1), \Sigma(x_0, w, t_1 - \tau))\right| \\ &\quad + \left|b(t_1, \Sigma(x_0, w, t_1), \Sigma(x_0, w, t_1 - \tau)) - b(t_2, \Sigma(x_0, w, t_2), \Sigma(x_0, w, t_2 - \tau))\right| \\ &\quad + \left|b(t_2, \Sigma(x_0, w, t_2), \Sigma(x_0, w, t_2 - \tau)) - b(t_2, \Sigma(x_0, w_{i_k}, t_2), \Sigma(x_0, w_{i_k}, t_2 - \tau))\right| \\ &\leq \delta'/4 + \delta'/4 + \delta'/4 = 3\delta'/4 \end{aligned}$$

for all $t_1, t_2 \in [t_a, t']$ satisfying $|t_1 - t_2| < \beta$.

Similarly,

$$\left|c(t_1, \Sigma(x_0, w_{i_k}, t_1), \Sigma(x_0, w_{i_k}, t_1 - \tau)) - c(t_2, \Sigma(x_0, w_{i_k}, t_2), \Sigma(x_0, w_{i_k}, t_2 - \tau))\right| < 3\delta'/4$$

for all $t_1, t_2 \in [t_a, t']$ satisfying $|t_1 - t_2| < \beta$. As $\beta > 0$, the last two inequalities rule out the possibility of $\beta^*(\varepsilon') = 0$ for all $\varepsilon' > 0$.

Now, consider another member $\Sigma' \in \mathcal{F}_\gamma(\Sigma_0)$. In (3.15), substitute t' for t_c and $\delta'/[8(\alpha+\gamma)M]$ for γ. Combining this with (2.8), we get

$$\begin{aligned}
&\big|b(t_1, \Sigma'(x_0, w_{i_k}, t_1), \Sigma'(x_0, w_{i_k}, t_1-\tau)) \\
&\quad -b(t_2, \Sigma'(x_0, w_{i_k}, t_2), \Sigma'(x_0, w_{i_k}, t_2-\tau))\big| \\
&\le \big|b(t_1, \Sigma'(x_0, w_{i_k}, t_1), \Sigma'(x_0, w_{i_k}, t_1-\tau)) \\
&\quad -b(t_1, \Sigma(x_0, w_{i_k}, t_1), \Sigma(x_0, w_{i_k}, t_1-\tau))\big| \\
&\quad +\big|b(t_1, \Sigma(x_0, w_{i_k}, t_1), \Sigma(x_0, w_{i_k}, t_1-\tau)) \\
&\quad -b(t_2, \Sigma(x_0, w_{i_k}, t_2), \Sigma(x_0, w_{i_k}, t_2-\tau))\big| \\
&\quad +\big|b(t_2, \Sigma'(x_0, w_{i_k}, t_2), \Sigma'(x_0, w_{i_k}, t_2-\tau)) \\
&\quad -b(t_2, \Sigma(x_0, w_{i_k}, t_2), \Sigma(x_0, w_{i_k}, t_2-\tau))\big| \\
&\le (\alpha+\gamma)\frac{\delta'}{8(\alpha+\gamma)} + 3\delta'/4 + (\alpha+\gamma)\frac{\delta'}{8(\alpha+\gamma)} \\
&= \delta'/4 + 3\delta'/4 = \delta'
\end{aligned}$$

for all $|t_1 - t_2| < \beta$. By choosing δ' sufficiently small, we get

$$\begin{aligned}
&\big|b(t_1, \Sigma'(x_0, w_{i_k}, t_1), \Sigma'(x_0, w_{i_k}, t_1-\tau)) \\
&\quad -b(t_2, \Sigma'(x_0, w_{i_k}, t_2), \Sigma'(x_0, w_{i_k}, t_2-\tau))\big| < \varepsilon
\end{aligned}$$

for all $|t_1 - t_2| < \beta$ and $k \ge N$. Similarly,

$$\begin{aligned}
&\big|c(t_1, \Sigma'(x_0, w_{i_k}, t_1), \Sigma'(x_0, w_{i_k}, t_1-\tau)) \\
&\quad -c(t_2, \Sigma'(x_0, w_{i_k}, t_2), \Sigma'(x_0, w_{i_k}, t_2-\tau))\big| < \varepsilon
\end{aligned}$$

for all $|t_1 - t_2| < \beta$ and all integers $k \ge N$. Since $\beta > 0$ and this holds for all members $\Sigma' \in \mathcal{F}_\gamma(\Sigma_0)$, our proof concludes. ■

We proceed now to prove Theorem 5.2.

Proof of Theorem 5.2: As the residual input v and the initial condition x_0 are the same for all members of $\mathcal{F}_\gamma(\Sigma_0)$, we have

$$x(t) = x^\pm(t) \text{ for all } t \le 0. \tag{5.7}$$

Consider two times $t_1, t_2 \in [0, t']$, $t_1 < t_2$, and let $\lambda > 0$ be a real number for which the ratio $p := (t_2 - t_1)/\lambda$ is an integer. Construct the partition

$$\begin{aligned}
&[t_1, t_2] \\
&= \big\{[t_1, t_1+\lambda], [t_1+\lambda, t_1+2\lambda], \dots, [t_1+(p-1)\lambda, t_2]\big\}.
\end{aligned} \tag{5.8}$$

In each sub-interval of this partition, we select below m points $\theta_1^q, \theta_2^q, \dots, \theta_m^q \in [t_1 + q\lambda, t_1 + (q+1)\lambda]$, $q = 0, 1, \dots, p-1$, to serve as switching times of a bang-bang control input signal $u^\pm = \big(u_1^\pm, \dots, u_m^\pm\big)^T \in U(K)$ given by

$$u_i^\pm(t) := \begin{cases} +K & \text{for } t \in [t_1 + q\lambda, \theta_i^q), \\ -K & \text{for } t \in [\theta_i^q, t_1 + (q+1)\lambda), \\ & \quad \text{if } \theta_i^q < t_1 + (q+1)\lambda, \end{cases} \tag{5.9}$$

$i = 1, 2, \dots, m$, $q = 0, 1, \dots, p-1$.

The point θ_i^q is determined by the given control input signal $u = (u_1, u_2, \dots, u_m)^T \in U(K)$ from the equation

$$K\big[2(\theta_i^q - (t_1 + q\lambda)) - \lambda\big] = \int_{t_1+q\lambda}^{t_1+(q+1)\lambda} u_i(s)ds. \tag{5.10}$$

The fact that $-K\lambda \le \int_{t_1+q\lambda}^{t_1+(q+1)\lambda} u_i(s)ds \le K\lambda$ implies that there is a unique solution θ_i^q of (5.10) for each $i \in \{1, 2, \dots, m\}$ and each $q \in \{0, 1, 2, \dots, p-1\}$. With these $\{\theta_i^q\}$, the bang-bang signal $u^\pm$ of (5.9) satisfies

$$\begin{aligned}
&\int_{t_1+q\lambda}^{t_1+(q+1)\lambda} \big(u_i(s) - u_i^\pm(s)\big)\, ds = 0,\ i \in \{1, 2, \dots, m\}, \\
&q \in \big\{0, 1, 2, \dots, p-1\big\}.
\end{aligned} \tag{5.11}$$

Using the activation time t_a of Convention 2.1, consider the combined signals

$$w(t) := \begin{cases} v(t) & t \in [t_a, 0), \\ u(t) & t \ge 0; \end{cases}$$

$$w^\pm(t) := \begin{cases} v(t) & t \in [t_a, 0), \\ u^\pm(t) & t \ge 0. \end{cases}$$

As the same residual input signal is used in both cases, we have

$$\int_{\theta'}^{\theta''} \big(w_i(s) - w_i^\pm(s)\big)\, ds = 0 \text{ for all } \theta', \theta'' \le 0.$$

Examine now the difference $\xi(t) := x(t) - x^\pm(t)$, $t \in [t_a, t']$. By (5.7), we have $\xi(t) = 0$ for all $t \le 0$. Invoking (2.4) at a time $t \in (t_1, t_2]$, we get

$$\begin{aligned}
\xi(t) = \xi(t_1) + \int_{t_1}^{t} \big[&a(s, x(s), x(s-\tau)) \\
&- a(s, x^\pm(s), x^\pm(s-\tau)) \\
&+ b(s, x(s), x(s-\tau))w(s) \\
&- b(s, x^\pm(s), x^\pm(s-\tau))w^\pm(s) \\
&+ c(s, x(s), x(s-\tau))w(s-\tau) \\
&- c(s, x^\pm(s), x^\pm(s-\tau))w^\pm(s-\tau)\big]ds.
\end{aligned}$$

Using (2.6) with the bounds (2.7), (2.8), $|w| \le K$, and $|w^{\pm}| \le K$, yields

$$\begin{aligned}\sup_{t\in[t_1,t_2]} |\xi(t)| &\le |\xi(t_1)| + (\alpha+\gamma)\sup_{s\in[t_1-\tau,t_2]}|\xi(s)|(t_2-t_1)\\ &+(\alpha+\gamma)(t_2-t_1)\sup_{s\in[t_1-\tau,t_2]}|\xi(s)|K\\ &+\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t} b(s,x(s),x(s-\tau))\big(w(s)-w^{\pm}(s)\big)\,ds\right|\\ &+(\alpha+\gamma)(t_2-t_1)\sup_{s\in[t_1-\tau,t_2]}|\xi(s)|K\\ &+\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t} c(s,x(s),x(s-\tau))\big(w(s-\tau)-w^{\pm}(s-\tau)\big)\,ds\right|.\end{aligned}$$

Collecting terms, using the facts that $\sup_{s\in[t_1-\tau,t_2]}|\xi(s)| \le \sup_{t\in[t_1-\tau,t_1]}|\xi(t)| + \sup_{t\in[t_1,t_2]}|\xi(t)|$ and $|\xi(t_1)| \le \sup_{t\in[t_1-\tau,t_1]}|\xi(t)|$, and shifting the integration variable of the second integral, we obtain

$$\begin{aligned}&[1-(\alpha+\gamma)(1+2K)(t_2-t_1)]\sup_{t\in[t_1,t_2]}|\xi(t)|\\ &\le [1+(\alpha+\gamma)(1+2K)(t_2-t_1)]\sup_{t\in[t_1-\tau,t_1]}|\xi(t)|\\ &\quad+\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t} b(s,x(s),x(s-\tau))\big(w(s)-w^{\pm}(s)\big)\,ds\right|\\ &\quad+\sup_{t\in[t_1,t_2]}\left|\int_{t_1-\tau}^{t-\tau} c(s+\tau,x(s+\tau),x(s))\big(w(s)-w^{\pm}(s)\big)\,ds\right|.\end{aligned} \tag{5.12}$$

Now, select a real number $\mu \in (0, t'-t_1]$ such that $(\alpha+\gamma)(1+2K)\mu < 1$, set

$$t_2 := t_1 + \mu, \tag{5.13}$$

and denote

$$\eta_1 := \frac{1+(\alpha+\gamma)(1+2K)\mu}{1-(\alpha+\gamma)(1+2K)\mu},\qquad \eta_2 := \frac{1}{1-(\alpha+\gamma)(1+2K)\mu}.$$

Then, (5.12) yields

$$\begin{aligned}\sup_{t\in[t_1,t_1+\mu]}|\xi(t)| &\le \eta_1\sup_{t\in[t_1-\tau,t_1]}|\xi(t)|\\ &+\eta_2\sup_{t\in[t_1,t_1+\mu]}\left|\int_{t_1}^{t} b(s,x(s),x(s-\tau))\big(w(s)-w^{\pm}(s)\big)\,ds\right|\\ &+\eta_2\sup_{t\in[t_1,t_1+\mu]}\left|\int_{t_1-\tau}^{t-\tau} c(s+\tau,x(s+\tau),x(s))\big(w(s)-w^{\pm}(s)\big)\,ds\right|.\end{aligned} \tag{5.14}$$

To estimate the last two integral terms, use the partition (5.8) with $t_2 = t_1 + \mu$, and let $q(t) \in \{0, 1, 2, \dots, p-1\}$ be such that $t \in [q(t)\lambda, (q(t)+1)\lambda]$. Note that, since $t_1 \ge 0$, we have $w(t) = u(t)$ for $t \in [t_1, t_1+\mu]$. Then,

$$\begin{aligned}&\sup_{t\in[t_1,t_1+\mu]}\left|\int_{t_1}^{t} b(s,x(s),x(s-\tau))\big(w(s)-w^{\pm}(s)\big)\,ds\right|\\ &=\sup_{t\in[t_1,t_1+\mu]}\left|\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda} b(s,x(s),x(s-\tau))\big(u(s)-u^{\pm}(s)\big)\,ds\right.\\ &\quad\left.+\int_{t_1+q(t)\lambda}^{t} b(s,x(s),x(s-\tau))\big(u(s)-u^{\pm}(s)\big)\,ds\right|\\ &=\sup_{t\in[t_1,t_1+\mu]}\left|\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\Big\{b(t_1+i\lambda,x(t_1+i\lambda),x(t_1-\tau+i\lambda))\right.\\ &\quad-b(t_1+i\lambda,x(t_1+i\lambda),x(t_1-\tau+i\lambda))+b(s,x(s)),x(s-\tau)\Big\}\\ &\quad\left.\times\big(u(s)-u^{\pm}(s)\big)\,ds+\int_{t_1+q(t)\lambda}^{t} b(s,x(s),x(s-\tau))\big(u(s)-u^{\pm}(s)\big)\,ds\right|\\ &\le\sup_{t\in[t_1,t_1+\mu]}\left|\sum_{i=0}^{q(t)-1} b(t_1+i\lambda,x(t_1+i\lambda),x(t_1-\tau+i\lambda))\right.\\ &\quad\left.\times\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\big(u(s)-u^{\pm}(s)\big)\,ds\right|\\ &\quad+\sup_{t\in[t_1,t_1+\mu]}\left|\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\big[b(s,x(s),x(s-\tau))\right.\\ &\quad\left.-b(t_1+i\lambda,x(t_1+i\lambda),x(s-\tau+i\lambda))\big]\big(u(s)-u^{\pm}(s)\big)\,ds\right|\\ &\quad+\sup_{t\in[t_1,t_1+\mu]}\left|\int_{t_1+q(t)\lambda}^{t} b(s,x(s),x(s-\tau))\big(u(s)-u^{\pm}(s)\big)\,ds\right|.\end{aligned}$$

Choose a real number $\varepsilon > 0$ and, in the notation of Lemma 5.1, set $\lambda \le \beta(t',\varepsilon)$. Then, using (5.11), the bound $B(t')$ of Corollary 3.1, and the fact that $0 \le p\lambda = t_2 - t_1 = \mu$, we get

$$\begin{aligned}&\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t} b(s,x(s),x(s-\tau))\big(u(s)-u^{\pm}(s)\big)\,ds\right|\\ &\le 2K\varepsilon\mu + 2KB(t')\lambda.\end{aligned} \tag{5.15}$$

Similarly,

$$\begin{aligned}&\sup_{t\in[t_1,t_2]}\left|\int_{t_1-\tau}^{t-\tau} c(s+\tau,x(s+\tau),x(s))\big(w(s)-w^{\pm}(s)\big)\,ds\right|\\ &\le 2K\varepsilon\mu + 2KB(t')\lambda.\end{aligned} \tag{5.16}$$

Substituting (5.15) and (5.16) into (5.14), we obtain

$$\sup_{t\in[t_1,t_2]}|\xi(t)| \le \eta_1\sup_{t\in[t_1-\tau,t_1]}|\xi(t)| + 4K\eta_2\big[\varepsilon\mu + B(t')\lambda\big]. \tag{5.17}$$

Now, let $\delta > 0$ be a real number. Select $\varepsilon > 0$ so that $4K\eta_2\varepsilon\mu < \delta/2$ and select $\lambda > 0$ so that $\lambda \le \beta(t',\varepsilon)$ and $4K\eta_2 B(t')\lambda < \delta/2$. Then, (5.17) becomes

$$\sup_{t\in[t_1,t_1+\mu]}|\xi(t)| \le \delta + \eta_1\sup_{t\in[t_1-\tau,t_1]}|\xi(t)|,$$

which resembles (3.2); denote

$$\zeta_k := \sup_{t\in[t_a+(k-1)\mu, t_a+k\mu]} |\xi(t)|.$$

Adapting to the present case, the argument leading from (3.2) to (3.8) yields $|\zeta_k| \le d_k\delta, k = 1, 2, \dots$. Continuing to use the notation used after (3.8), let r be an integer satisfying $r \ge (t' - t_a)/\mu$, and set $D := \max_{i=1,2,\dots,r} d_k$. Then, $\sup_{t\in[t_a,t']} |\xi(t)| \le D\delta$, and the theorem follows by selecting $\delta < \sigma/D$. This concludes our proof. ■

We turn now to the proof of Theorem 5.1.

Proof of Theorem 5.1: By Theorem 4.1, there is a minimal time $t^*(x_0, \gamma, \ell)$ and an optimal control input signal $u^*(x_0, \gamma, \ell) \in U(K)$ satisfying $t^*(x_0, \gamma, \ell) = t(x_0, \gamma, \ell, u^*(x_0, \gamma, \ell))$. Now, let $\sigma > 0$ be a real number. By Theorem 5.2, there is a bang-bang control input signal $u^\pm \in U(K)$ with a finite number of switchings for which $|\Sigma(x_0, v, u^*(x_0, \gamma, \ell), t) - \Sigma(x_0, v, u^\pm, t)| < \sigma$ for all $t \in [0, t^*(x_0, \gamma, \ell)]$, for all $v \in V_\gamma(v_0)$, and for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.

Next, note that any vectors $y, z \in R^n$ satisfy $z^Tz = y^Ty - 2y^T(y - z) + (y - z)^T(y - z) \le y^Ty + 2n|y||y - z| + n|y - z|^2$. Thus, since $\Sigma(x_0, v, u^*(x_0, \gamma, \ell), t^*(x_0, \gamma, \ell)) \in \rho(\ell)$, this yields

$$\begin{aligned}
&\Sigma^T(x_0, v, u^\pm, t^*(x_0, \gamma, \ell))\Sigma(x_0, v, u^\pm, t^*(x_0, \gamma, \ell)) \\
&\quad \le \Sigma^T(x_0, v, u^*(x_0, \gamma, \ell), t^*(x_0, \gamma, \ell)) \\
&\qquad \times \Sigma(x_0, v, u^*(x_0, \gamma, \ell), t^*(x_0, \gamma, \ell)) \\
&\qquad + 2n\left|\Sigma(x_0, v, u^*(x_0, \gamma, \ell), t^*(x_0, \gamma, \ell))\right| \\
&\qquad \times \left|\Sigma(x_0, v, u^*(x_0, \gamma, \ell), t^*(x_0, \gamma, \ell)) \right. \\
&\qquad \left. - \Sigma(x_0, v, u^\pm, t^*(x_0, \gamma, \ell))\right| \\
&\qquad + n\left|\Sigma(x_0, v, u^*(x_0, \gamma, \ell), t^*(x_0, \gamma, \ell)) \right. \\
&\qquad \left. - \Sigma(x_0, v, u^\pm, t^*(x_0, \gamma, \ell))\right|^2 \\
&\quad \le \ell + 2n\sqrt{\ell}\sigma + n\sigma^2.
\end{aligned}$$

As $\ell' - \ell > 0$, we can select σ to satisfy $2n\sqrt{\ell}\sigma + n\sigma^2 \le \ell' - \ell$. Then, $\Sigma(x_0, v, u^\pm, t^*(x_0, \gamma, \ell)) \in \rho(\ell')$, and it follows that $t(x_0, \gamma, \ell', u^\pm) \le t^*(x_0, \gamma, \ell)$. This concludes our proof. ■

In brief terms, Theorem 5.1 shows that optimal performance can be approximated as closely as desired by using bang-bang control input signals. Considering that it is easier to work with bang-bang signals than with general vector-valued signals, this fact simplifies the calculation and the implementation of controllers that reduce operating errors in minimal time.

6. Example

Consider the following nonlinear system:

$$\begin{aligned}
\Sigma : \dot{x}_1(t) &= (1 + d_1 \sin t)x_2(t) + d_2x_2(t - 1) \\
&\quad + (2 + d_3 \cos t)u(t - 1), \\
\dot{x}_2(t) &= d_4 \cos 2t \frac{x_1^2(t)}{1 + x_1^2(t)} + x_1(t - 1) \\
&\quad + (2 + \cos x_2(t))u(t - 1), \qquad (6.1)
\end{aligned}$$

where d_1, d_2, d_3, and d_4 are constant parameters with unspecified values in the ranges $0.3 \le d_1, d_2 \le 0.5$ and $0.8 \le d_3, d_4 \le 1$; the nominal values are $d_1^0, d_2^0 = 0.4$ and $d_3^0, d_4^0 = 0.9$. Here, the delay time is $\tau = 1$, and the uncertainty parameter is $\gamma = 0.1$. The initial state of the system is $x_0 = [-1, 4]^T$, and the residual input signal is a constant signal with unspecified value in the range $v(t) \in [-0.1, 0.1]$; the nominal value is $v_0(t) = 0, t \le 0$. The input signal bound is $K = 5$, and the operating error bound is $\ell = 1$.

We use a numerical optimisation process to calculate a bang-bang control input signal $u^\pm(t)$ that approximates optimal performance. Often, $u^\pm(t)$ can be derived via a numerical search process, as described briefly in Remark 6.1.

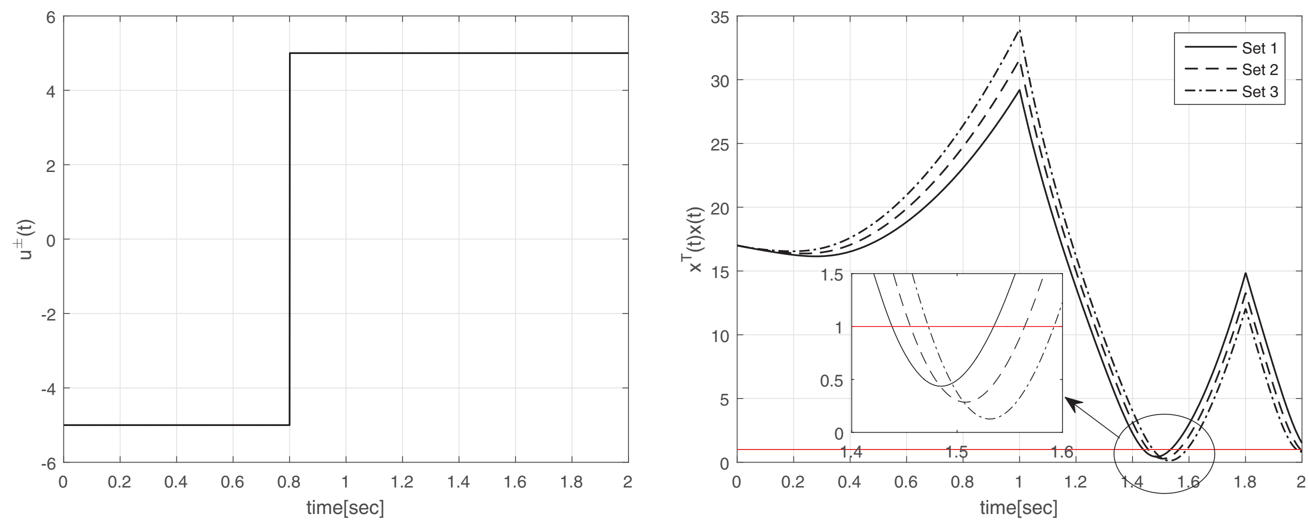
In the present case, numerical optimisation shows that the minimal time $t^*(x_0, \gamma, \ell)$ is about 1.47 seconds. A numerical search process (see Remark 6.1) shows that the simple bang-bang signal $u^\pm(t)$ of Figure 2(A) achieves almost optimal performance, as can be seen in Figure 2(B). The plots of Figure 2(B) show the performance for three sets of parameters:

$$\begin{aligned}
&\text{Set 1:}\{v(t) = -0.1, d_1 = 0.5, d_2 = 0.5, d_3 = 1, d_4 = 1\}; \\
&\text{Set 2:}\{v(t) = 0, d_1 = 0.4, d_2 = 0.4, d_3 = 0.9, d_4 = 0.9\}; \\
&\text{Set 3:}\{v(t) = 0.1, d_1 = 0.3, d_2 = 0.3, d_3 = 0.8, d_4 = 0.8\} \qquad (6.2)
\end{aligned}$$

For comparison, consider the case where the control input signal $u(t)$ is a constant. A constant input signal is used in the sample-and-hold method commonly employed in sampled-data control systems. For the present example, simulation shows that there is no constant input signal $u \in [-5, 5]$ that takes all members of $\mathcal{F}_\gamma(\Sigma_0)$ from x_0 to the domain $\rho(1)$. Thus, a constant control input signal cannot resolve the control problem in this case, although a solution does exist.

Remark 6.1 Design of the controller *C*: The bang-bang control input signal $u^\pm(t)$ of Figure 2(A) was derived through a numerical search process, as follows.

Estimate a bound of the minimal time: qualitative considerations based on the structure of Equation (6.1), the



(A) Bang-bang input signal

(B) Operating error progression

Figure 2. Demonstration.

input signal bound K, and initial state x_0 lead to the conclusion that $t^*(x_0, \gamma, \ell) \leq 3$. Thus, interest is focused on the time interval $[0, 3]$.

Create the partition P: partition the time interval $[0, 3]$ into 300 equal subintervals of duration 0.01 each. The endpoints of the partition's subintervals serve as potential switching times for bang-bang input signals.

Choose representatives of the family $\mathcal{F}_\gamma(\Sigma_0)$: select a class of N representatives $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_N\}$ of the family $\mathcal{F}_\gamma(\Sigma_0)$ and its residual input signals; the systems of (6.2) can form part of such a class.

Generate bang-bang input signals: for an integer $j \in \{0, 1, \ldots, 300\}$, let U_j be the family of all bang-bang signals with j switching times at points of the partition P.

The minimal time: set

$$t_j^{\pm} := \min_{u \in U_j}\{t : \max_{i=1,2,\ldots,N} \Sigma_i^T(x_0, v_i, u, t)\Sigma_i(x_0, v_i, u, t) \leq \ell\}.$$

Terminate: stop the process at step j if $t_j^{\pm} - t_{j+1}^{\pm} < \varepsilon$, where $\varepsilon > 0$ is an acceptable tolerance. The member of U_j corresponding to $t_j^{\pm}$ is then an appropriate bang-bang control input signal.

If necessary, more advanced numerical optimisation techniques can be employed.

7. Conclusion

In this paper, we considered the design of optimal controllers that reduce operating errors as quickly as possible after a feedback disruption. We showed that such optimal controllers do exist under rather general conditions. We also showed that optimal performance can be approximated as closely as desired by controllers that generate bang-bang signals. The use of bang-bang signals provides a relatively simple path toward designing and implementing controllers that approximate optimal performance.

Feedback disruptions abound in modern control systems. One common situation where feedback disruptions are inevitable is in the control of continuous-time systems by digital controllers. In this application, sampling is used to transmit the controlled system's response to the controller; needless to say, feedback disruptions occur between samples. Controllers derived in this paper can help reduce the resulting operating errors quickly, once the next sample has arrived.

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