Factorization of Linear Systems: A Generalized Framework*

Jacob Hammer[1]
Department of Mathematics
Case Western Reserve University
Cleveland, Ohio 44106

and

Michael Heymann[‡]
Department of Electrical Engineering
Technion – Israel Institute of Technology
Haifa, Israel

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ABSTRACT

A generalized module theoretic framework for the study of linear time invariant systems is developed. Crucial in the present discussion are two new notions: the generalized "order" and the "adapted bases". These notions form generalizations of the classical concepts of order (or degree) and of proper bases, employed in the theory of linear systems. The resulting framework is then applied to obtain explicit conditions for system factorization, and to study output feedback systems in which the feedback compensator is stable.

1. INTRODUCTION

In the present paper we develop a unified module theoretic framework for the investigation of linear time invariant systems. The main purpose is to show that the theories of realization (Kalman [10]), of state feedback (Hautus and Heymann [6]), of causality (Hammer and Heymann [5]), of strict observability (Hammer and Heymann [4]), and of stability (Hammer [2]), previously studied, can be regarded as different manifestations of a uniform

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underlying algebraic framework. It will be convenient to start with a brief review of the point of view adopted in the abovementioned works.

Let K be a field, and let Σ be a linear time invariant system, admitting inputs from the finite dimensional K-linear space U and having its outputs in the finite dimensional K-linear space Y. For the sake of intuitive convenience, we assume that Σ is a discrete time system. Every input sequence to Σ can then be regarded as a formal Laurent series $u = \sum_{t=t_0}^{\infty} u_t z^{-t}$, where t is the time marker, $t_0 \neq -\infty$, and $u_t \in U$ for all t. The set of all such formal Laurent series (where t_0 is allowed to range over all integers, and (u_t) over all U) is denoted by ΛU. Thus, every input sequence to Σ is an element in ΛU. Similarly, every output sequence from Σ is an element in ΛY, so that Σ induces a map $\bar{f}: \Lambda U \to \Lambda Y$.

The employment of the sets ΛU and ΛY is motivated by certain algebraic properties that they possess. In particular, it can be shown that the set ΛK (of formal Laurent series with coefficients in the base field K) is endowed with a field structure under the operations of coefficientwise addition and sequence convolution as multiplication. Under similar operations, the set ΛU becomes a linear space over the field ΛK, and moreover, $\dim_{\Lambda K} \Lambda U = \dim_K U$. The importance of these observations stems from the fact that ΛK-linearity is closely related to time invariance [11, Chapter 10; 16]. Indeed, when the map $\bar{f}: \Lambda U \to \Lambda Y$ induced by Σ is ΛK-linear, then $\bar{f}zu = z\bar{f}u$ for all $u \in \Lambda U$, and the commutativity of $\bar{f}$ with the shift operator implies the time invariance. Conversely, under a mild assumption on Σ [5], it is also true that, when Σ is time invariant, the map $\bar{f}: \Lambda U \to \Lambda Y$ induced by Σ is ΛK-linear. Thus, ΛK-linear spaces form a natural algebraic framework for the study of linear time invariant systems. Throughout our discussion we shall limit ourselves and consider only ΛK-linear maps $\bar{f}: \Lambda U \to \Lambda Y$, where U and Y are finite dimensional K-linear spaces, and we shall denote

$$m := \dim_K U, \qquad p := \dim_K Y.$$

A ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ can, of course, be represented as a matrix, relative to specified bases $u_1, \dots, u_m \in \Lambda U$ and $y_1, \dots, y_p \in \Lambda Y$. Of particular importance is the case when the elements $u_1, \dots, u_m$ belong to U and the elements $y_1, \dots, y_p$ belong to Y (where U is regarded as a subset of ΛU, and Y is regarded as a subset of ΛY). In this case the matrix representation of $\bar{f}$ is called a *transfer matrix* and it coincides with the classical notion of transfer matrices. In our discussion below, whenever considering matrix representations, we shall always assume that they are transfer matrices. For the sake of conciseness, we shall make no distinction between a map and its transfer matrix.

A ΛK-linear map can also be regarded as an element in a certain space of Laurent series, as follows. Let U and Y be finite dimensional K-linear spaces, and let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map. As is well known, the set of all K-linear maps $U \to Y$ forms a K-linear space, which we denote by L. Similarly, the set of all ΛK-linear maps $\Lambda U \to \Lambda Y$ forms a ΛK-linear space, which we denote by $\mathcal{L}$. The point is that $\mathcal{L}$ can be identified with the ΛK-linear space of Laurent series ΛL, as follows [6]. With each element $T = \sum T_t z^{-t} \in \Lambda L$, we associate a ΛK-linear map $\bar{f}_T: \Lambda U \to \Lambda Y$, which maps an element $u = \sum u_t z^{-t} \in \Lambda U$ into

$$\bar{f}_T u = \sum_t \left(\sum_k T_k u_{t-k} \right) z^{-t}.$$

Conversely, let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map, and define the K-linear maps

$$i_u: U \to \Lambda U: u \mapsto u \qquad \text{(canonical injection)},$$

$$p_k: \Lambda Y \to Y: \sum y_t z^{-t} \mapsto y_k.$$

Then, we associate with $\bar{f} \in \mathcal{L}$ the element $T_{\bar{f}} = \sum T_t z^{-t} \in \Lambda L$, where $T_t = p_t \bar{f} i_u$ for all t. It can be readily seen that $T_{\bar{f}_T} = T$ and that $\bar{f}_{T_{\bar{f}}} = \bar{f}$. The element $T_{\bar{f}} \in \Lambda L$ is called the *transfer function* of $\bar{f}$ (and it is to be distinguished from the transfer *matrix* defined above).

We review now a few facts regarding the structure of ΛK-linear spaces. Let S be a K-linear space. The set ΛS contains, as subsets, the set $\Omega^+ S$ of all (polynomial) elements of the form $\sum_{t=t_0}^{0} s_t z^{-t}$, $t_0 \leqslant 0$, and the set $\Omega^- S$ of all (power series) elements of the form $\sum_{t=0}^{\infty} s_t z^{-t}$. In particular, it is well known that the sets $\Omega^+ K$ and $\Omega^- K$ form principal ideal domains under the operations defined in ΛK. The sets $\Omega^+ S$ and $\Omega^- S$ are then $\Omega^+ K$- and $\Omega^- K$-modules, respectively, and $\operatorname{rank}_{\Omega^+ K} \Omega^+ S = \operatorname{rank}_{\Omega^- K} \Omega^- S = \dim_K S$.

A ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is *polynomial* if it can be restricted to the set of polynomials, namely, if $\bar{f}[\Omega^+ U] \subset \Omega^+ Y$. Equivalently, $\bar{f}$ is polynomial if and only if all the entries in its transfer matrix are in $\Omega^+ K$. A ΛK-linear map is called *rational* if there exists a nonzero polynomial $\psi \in \Omega^- K$ such that $\psi \bar{f}$ is a polynomial map. Analogously, a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is *causal* (respectively, *strictly causal*) if $\bar{f}[\Omega^- U] \subset \Omega^- Y$ (respectively, $\bar{f}[\Omega^- U] \subset z^{-1}\Omega^- Y$). Equivalently, $\bar{f}$ is causal (respectively, strictly causal) if and only if all entries in its transfer matrix belong to $\Omega^- K$ (respectively, $z^{-1}\Omega^- K$). A ΛK-linear map which is both strictly causal and rational is called a *linear i/o (input/output) map*. Finally, a ΛK-linear map $\bar{l}: \Lambda U \to \Lambda U$ is *bicausal* if it is causal and has a causal inverse.

2. RATIONALITY AND STABILITY: GENERAL CONSIDERATIONS

Let S be a finite dimensional K-linear space. An element $s \in \Lambda S$ is called $\Omega^+ K$*-rational* (or sometimes simply *rational*) if there exists a nonzero polynomial $\psi \in \Omega^+ K$ such that $\psi s \in \Omega^+ S$. The set of $\Omega^+ K$-rationals in ΛS is denoted $Q_{\Omega^+ K} S$. For an element $s \in Q_{\Omega^+ K} S$, the set of polynomials $\psi \in \Omega^+ K$ for which $\psi s \in \Omega^+ S$ is easily seen to be an ideal in $\Omega^+ K$. Since $\Omega^+ K$ is a principal ideal domain, this ideal is generated by a monic polynomial ψ_s, which we call the *least denominator* of s. The zeros of ψ_s are called the *poles* of s. (In case $K = \mathbb{R}$, the field of real numbers, it is customary to consider not only poles in $\mathbb{R}$ but also in $\mathbf{C}$, the field of complex numbers). The present definition of $\Omega^+ K$-rationality applies, in particular, also to transfer functions of ΛK-linear maps, and we call a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ $\Omega^+ K$*-rational* (or, simply, *rational*) if its transfer function $T_{\bar{f}} \in \Lambda L$ is. This definition is clearly consistent with the definition of rational ΛK-linear maps given in Section 1.

We turn now to the concept of stability. If $\mathfrak{D}$ is a set of polynomials, we say that an $\Omega^+ K$-rational map is $\mathfrak{D}$*-stable* if its least denominator is in $\mathfrak{D}$. In order to ensure that the set of $\mathfrak{D}$-stable maps has convenient mathematical properties, a number of restrictions on the set $\mathfrak{D}$ are required [13].

DEFINITION 2.1. A set $\mathfrak{D}$ of (monic) polynomials over K is called a *denominator set* if it satisfies the following conditions:

(i) $\mathfrak{D}$ is *multiplicatively closed*, i.e., $p \in \mathfrak{D}$, $q \in \mathfrak{D}$ imply $p \cdot q \in \mathfrak{D}$.
(ii) The unit polynomial 1 belongs to $\mathfrak{D}$, but the zero polynomial does not.
(iii) $\mathfrak{D}$ contains at least one polynomial of degree one, i.e., there exists $\alpha \in K$ such that $z - \alpha \in \mathfrak{D}$.
(iv) $\mathfrak{D}$ is *saturated*, i.e., if $p \in \mathfrak{D}$ and q is a monic divisor of p, then $q \in \mathfrak{D}$.

Conditions (i) and (ii) say that $\mathfrak{D}$ is a *multiplicative set* (see e.g., [17]), so that one can define the set $\Omega_{\mathfrak{D}} K$ as the set of fractions p/q where $p \in \Omega^+ K$ and $q \in \mathfrak{D}$. Conditions (iii) and (iv) are motivated by considerations that are discussed shortly. We need the following (see also [2]).

DEFINITION 2.2. Let $\mathfrak{D}$ be a denominator set and S a K-linear space. Then an element $s \in Q_{\Omega^+ K} S$ is called *stable* (or, explicitly, $\mathfrak{D}$*-stable*) if there exists $p \in \mathfrak{D}$ such that $ps \in \Omega^+ S$.

A ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is called *i/o (input/output) stable* (in the sense of $\mathfrak{D}$) if its transfer function $T_{\bar{f}} \in \Lambda L$ is $\mathfrak{D}$-stable.

We denote by $\Omega_{\mathfrak{D}} S$ the set of all $\mathfrak{D}$-stable elements in ΛS.

The above definition of stability is a generalization to arbitrary fields of the usual concept of stability in system theory defined in an algebraic framework. For example, in case $K = \mathbf{R}$, the field of real numbers, we let $\mathbf{C}^-$ be a prescribed subset of the complex plane satisfying $\mathbf{C}^- \cap \mathbf{R} \neq \emptyset$, and let $\mathfrak{D}$ be defined by

$$(2.3) \qquad \mathfrak{D} := \{p \in \Omega^+ K \mid p(z) = 0 \Rightarrow z \in \mathbf{C}^-\}.$$

Typical selections of $\mathbf{C}^-$ are $\mathbf{C}^- = \{z \in \mathbf{C} : |z| < 1\}$ in the discrete time case, and $\mathbf{C}^- = \{z \in \mathbf{C} : \operatorname{Re} z < 0\}$ in the continuous time case. The set $\mathfrak{D}$ defined by (2.3) satisfies conditions (i)–(iv) of Definition 2.1. In particular, condition (iii) corresponds to $\mathbf{C}^- \cap \mathbf{R} \neq \emptyset$. The following statement can be readily verified.

PROPOSITION 2.4. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map. Then $\bar{f}$ is i/o stable (in the sense of $\mathfrak{D}$) if and only if $\bar{f}[\Omega_{\mathfrak{D}} U] \subset \Omega_{\mathfrak{D}} Y$.*

The set $\Omega_{\mathfrak{D}} K$ is easily seen, by direct computation, to be a subring (with identity) of the rational field $Q_{\Omega^+ K}$ $(= Q_{\Omega^+ K} K)$. Actually, the following is true (see, e.g., Hammer [2], and Hautus and Sontag [9]).

PROPOSITION 2.5. *The ring $\Omega_{\mathfrak{D}} K$ is a principal ideal domain.*

Evidently, the set $\Omega_{\mathfrak{D}} S$ is an $\Omega_{\mathfrak{D}} K$-module, and it can be expressed in explicit terms in the following form. Let $s_1, \dots, s_m$ be any basis of the K-linear space S. Then

$$(2.6) \qquad \Omega_{\mathfrak{D}} S = \left\{ s \in \Lambda S \mid s = \sum_{i=1}^{m} \alpha_i s_i,\ \alpha_1, \dots, \alpha_m \in \Omega_{\mathfrak{D}} K \right\},$$

so that also $\operatorname{rank}_{\Omega_{\mathfrak{D}} K} \Omega_{\mathfrak{D}} S = \dim_K S$.

In many situations one is interested in the combination of causality and stability. The set $\Omega_{\mathfrak{D}}^- S$ of all elements in ΛS that are both causal and $\mathfrak{D}$-stable is given by the intersection

$$\Omega_{\mathfrak{D}}^- S = \Omega_{\mathfrak{D}} S \cap \Omega^- S.$$

In particular, it can be readily seen that the set $\Omega_{\mathfrak{D}}^- K$ forms a ring under the operations of addition and multiplication as defined in ΛK. Moreover, the following stronger result was proved by Morse [13].

PROPOSITION 2.7. *The ring $\Omega_{\bar{\wp}} K$ is a principal ideal domain.*

Again, we obtain that $\Omega_{\bar{\wp}} S$ is an $\Omega_{\bar{\wp}} K$-module, and that, for any basis $s_1, \ldots, s_m$ of the K-linear space S, explicit representation of $\Omega_{\bar{\wp}} S$ is given by

$$\Omega_{\bar{\wp}} S = \left\{ s \in \Lambda S \,|\, s = \sum_{i=1}^{m} \alpha_i s_i;\ \alpha_1, \ldots, \alpha_m \in \Omega_{\bar{\wp}} K \right\}. \tag{2.8}$$

Thus, we also have that $\operatorname{rank}_{\Omega_{\bar{\wp}} K} \Omega_{\bar{\wp}} S = \dim_K S$.

Summarizing our discussion up to this point, we have encountered the rings $\Omega^+ K$, $\Omega^- K$, $\Omega_{\wp} K$, and $\Omega_{\bar{\wp}} K$, all of which form principal ideal domains under the operations of addition and multiplication defined in ΛK. All of these rings play fundamental roles in the theory of linear time invariant systems, encompassing the aspects of realization, causality, and stability. As it turns out, from the algebraic point of view, the dominant property of these notably different rings happens to be the property they have in common, namely, the principal ideal domain property. It is therefore convenient to disregard all their other properties, and to concentrate on the study of principal ideal domains contained in ΛK. This is, basically, the main theme of our present discussion.

Let $\Omega K \subsetneq \Lambda K$ be a principal ideal domain (properly contained as a subring in ΛK), and let S be a finite dimensional K-linear space. The ΛK-linear space ΛS is then also an ΩK-module. Motivated by (2.6) and (2.8), we define ΩS to be the ΩK-submodule of ΛS generated by S, i.e., if $s_1, \ldots, s_m$ is a basis for S, then

$$\Omega S := \left\{ s \in \Lambda S \,|\, s = \sum_{i=1}^{m} \alpha_i s_i;\ \alpha_i \in \Omega K,\ i = 1, \ldots, m \right\}. \tag{2.9}$$

We shall make use of the following notation:

$$\begin{aligned} &j_{\Omega K} : \Omega S \to \Lambda S : s \mapsto s && \text{(natural injection)},\\ &\pi_{\Omega K} : \Lambda S \to \Lambda S / \Omega S =: \Gamma_{\Omega K} S && \text{(canonical projection)}. \end{aligned} \tag{2.10}$$

We extend now our terminology to the principal ideal domain ΩK. An element $s \in \Lambda S$ is called an *ΩK-element* if $s \in \Omega S$. Thus, a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is an *ΩK-map* in case its transfer function is an ΩK-element of ΛL. $\bar{f}$ is called *ΩK-unimodular* if it is an invertible ΩK-map and its inverse is also an ΩK-map. Clearly, a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is an ΩK-map if and only if all the entries in its transfer matrix belong to ΩK. An element $s \in \Lambda S$ is called *ΩK-rational* if there exists a nonzero element $\psi \in \Omega K$ such that $\psi s \in \Omega S$. The set of *ΩK-rationals* in ΛS is denoted $Q_{\Omega K} S$. Just as in the case of $\Omega^+ K$, the definition of ΩK-rationality also applies to transfer functions of ΛK-linear maps, and we call a ΛK-linear map *ΩK-rational* if its transfer function is. Thus, a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is ΩK-rational if and only if there exists a nonzero element $\psi \in \Omega K$ such that $\psi \bar{f}$ is an ΩK-map.

Intuitively, an ΩK-map is a ΛK-linear map that can be "restricted to Ω," as follows (the proof is by direct computation):

PROPOSITION 2.11. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map. Then, $\bar{f}$ is an ΩK-map if and only if $\bar{f}[\Omega U] \subset \Omega Y$.*

In similar terms we can also characterize ΩK-unimodular maps.

PROPOSITION 2.12. *A ΛK-linear map $\bar{l}: \Lambda U \to \Lambda U$ is ΩK-unimodular if and only if $\bar{l}[\Omega U] = \Omega U$ (or, equivalently, if and only if $\ker \pi_{\Omega K} \bar{l} = \Omega U$).*

3. THE ORDER AND ADAPTED BASES

In the present section we derive a finitary characterization of ΩK-maps. The underlying idea is to generalize the theory of proper bases, which plays a fundamental role in the finitary characterization of causal maps. We start with a brief review of the classical notions of order and proper bases. Let $s = \sum s_t z^{-t} \in \Lambda S$ be an element. The *order* of s is defined by

$$\text{(3.1)} \qquad \operatorname{ord} s := \begin{cases} \min_t (s_t \neq 0) & \text{if } s \neq 0, \\ \infty & \text{if } s = 0. \end{cases}$$

The *leading coefficient* $\hat{s}$ of s is defined as $\hat{s} := s_{\operatorname{ord} s}$ if $s \neq 0$, and $\hat{s} := 0$ if $s = 0$. A set of elements $s_1, \dots, s_n \in \Lambda S$ is *properly independent* if the leading coefficients $\hat{s}_1, \dots, \hat{s}_n$ are K-linearly independent [14, 15, 1]. A basis consisting of properly independent elements is called a proper basis. The following is an equivalent characterization of proper bases [1, 5].

PROPOSITION 3.2. *Let $s_1, \dots, s_n \in \Lambda S$ be a set of nonzero elements. Then $s_1, \dots, s_n$ are properly independent if and only if the following holds: For every set $\alpha_1, \dots, \alpha_n \in \Lambda K$,*

$$\operatorname{ord}\left(\sum_{i=1}^{n} \alpha_i s_i \right) = \min_{i=1,\dots,n} (\operatorname{ord} \alpha_i s_i).$$

The importance of proper bases is related to the fact that they allow a finitary characterization of causality, as follows [15, 4]:

THEOREM 3.3. *Let $u_1,\ldots,u_m$ be a proper basis of the ΛK-linear space ΛU, and let $\bar{f}\colon \Lambda U \to \Lambda Y$ be a ΛK-linear map. Then, $\bar{f}$ is causal if and only if* $\operatorname{ord} \bar{f}u_i \geqslant \operatorname{ord} u_i$ *for all $i = 1,\ldots,m$.*

We turn now to a generalization of these concepts to general principal ideal domains included in ΛK, starting with the generalization of the concept of order. As before, we let $\Omega K \subsetneq \Lambda K$ be a principal ideal domain properly contained as a subring in ΛK, and we let $Q_{\Omega K}$ denote the field of quotients generated by ΩK.

For an element $s \in \Lambda S$ we define the *ΩK-order* of s, denoted $\operatorname{ord}_{\Omega K} s$, as the set of all elements $\alpha \in Q_{\Omega K}$ for which $\alpha s \in \Omega S$, that is,

$$\operatorname{ord}_{\Omega K} s := \{\alpha \in Q_{\Omega K} \mid \alpha s \in \Omega S\}. \tag{3.4}$$

Whenever the underlying ring ΩK is fixed, we shall use the simpler notation $\operatorname{ord} s$ for $\operatorname{ord}_{\Omega K} s$. Clearly, when $s = 0$, we have that $\operatorname{ord}_{\Omega K} s = Q_{\Omega K}$, that is, the whole field of quotients. Further, we have the following

PROPOSITION 3.5. *Let $s \in \Lambda S$ be any element. Then* $\operatorname{ord} s \neq 0$ *if and only if $s \in Q_{\Omega K} S$.*

Proof. If $\operatorname{ord} s \neq 0$ there is an element $0 \neq \gamma = p/q \in \operatorname{ord} s$ (with $p, q \in \Omega K$) such that $(p/q)s \in \Omega S$, whence $ps \in \Omega S$ and s is ΩK-rational. Conversely, if $s \in Q_{\Omega K} S$, there exists $0 \neq p \in \Omega K$ such that $ps \in \Omega S$, whence $p \in \operatorname{ord} s$ and $\operatorname{ord} s \neq 0$. ■

It is easy to see that the set $\operatorname{ord}_{\Omega K} s$ is actually an ΩK-module (contained in $Q_{\Omega K}$). In fact, we have

PROPOSITION 3.6. *If $s \in \Lambda S$ is nonzero, then $\operatorname{ord}_{\Omega K} s$ is a cyclic ΩK-module.*

Proof. We shall prove only the cyclicity, i.e., that $\operatorname{ord} s$ is generated by a single element. Let $s \neq 0$, and consider the set $q(s) := \{\alpha s \mid \alpha \in \operatorname{ord} s\} \subset \Omega S$. Obviously $q(s)$ is an ΩK-module and, being a submodule of a finitely generated module over a P.I.D., it is finitely generated. Thus, there are elements $\alpha_1,\ldots,\alpha_m \in \operatorname{ord} s$ such that $\alpha_1 s,\ldots,\alpha_m s$ generate $q(s)$. Consequently, $\alpha_1,\ldots,\alpha_m$ generate $\operatorname{ord} s$. Let $\psi \in \Omega K$ be a common denominator of

$\alpha_1,\ldots,\alpha_m$ (i.e., $\alpha_i\psi \in \Omega K$, $i=1,\ldots,m$), and let (ψ^{-1}) $[\Omega K]$ denote the (cyclic) ΩK-module generated by ψ^{-1}. Then, clearly, $\alpha_i=(\alpha_i\psi)\psi^{-1}\in(\psi^{-1})[\Omega K]$, $i=1,\ldots,m$, whence $\operatorname{ord} s\subset(\psi^{-1})[\Omega K]$, implying that $\operatorname{ord} s$ is also cyclic as claimed. ■

Let $0\neq s\in\Lambda S$ be any element, and let $\alpha\in Q_{\Omega K}$ be any generator of $\operatorname{ord} s$ (possibly zero). If $\alpha'\in Q_{\Omega K}$ is another generator of $\operatorname{ord} s$, then, by Proposition 3.6, it is an associate of α with respect to ΩK, i.e., $\alpha'=\mu\alpha$ where $\mu\in\Omega K$ is a unit (i.e., an invertible). It follows that α is uniquely defined modulo units in ΩK, and it will sometimes be convenient to identify $\operatorname{ord} s$ with one of its generators.

We consider next several

EXAMPLES 3.7.

(i) *ΩK is the ring $\Omega^- K$ of causal elements.* In this case we have that $Q_{\Omega^- K}=\Lambda K$, since, for every $0\neq\alpha\in\Lambda K$, at least one of α,α^{-1} is in $\Omega^- K$. Thus, every element $s\in\Lambda S$ is $\Omega^- K$-rational. Now, let s be a nonzero element in ΛS. In view of Propositions 3.5 and 3.6, there is a nonzero element $\alpha\in\Lambda K$ such that $\operatorname{ord}_{\Omega K} s=\alpha[\Omega^- K]$. Since $\alpha\neq 0$, there are a unit μ of $\Omega^- K$ and an integer k such that $\alpha=\mu z^{-k}$. Hence, $\operatorname{ord}_{\Omega^- K} s=z^{-k}[\Omega^- K]$, and it can be readily seen that $k=\operatorname{ord} s$, where $\operatorname{ord} s$ is the order as defined in (3.1). Thus, $\operatorname{ord}_{\Omega^- K}$ essentially coincides with the classical notion of order (3.1).

(ii) *ΩK is the ring of polynomials $\Omega^+ K$.* In this case $Q_{\Omega^+ K}$ is the usual field of rationals. For an element $s\in\Lambda S$, $\operatorname{ord}_{\Omega^+ K} s\neq 0$ if and only if $s\in Q_{\Omega^+ K}S$, i.e., if and only if s is rational (in the classical sense). To compute the order explicitly, let $0\neq s\in Q_{\Omega^+ K}S$ be given as $s=(s_1,\ldots,s_m)$ with $s_i=p_i/q_i$, where $p_i,q_i\in\Omega^+ K$ are coprime for all $i=1,\ldots,m$. Then $\operatorname{ord}_{\Omega^+ K} s$ is generated by the rational element q/p, where q and p are the monic polynomials $q=\operatorname{lcm}(q_1,\ldots,q_m)$ and $p=\gcd(p_1,\ldots,p_m)$ (lcm and gcd denoting, respectively, the least common multiple and the greatest common divisor). To see this, write $p_i=p\bar{p}_i$ and $q=q_i\bar{q}_i$ for polynomials $\bar{p}_i,\bar{q}_i$, $i=1,\ldots,m$. Then

$$\begin{aligned}\frac{q}{p}s&=\left(\frac{q}{p}s_1,\ldots,\frac{q}{p}s_m\right)\\&=(\bar{q}_1\bar{p}_1,\ldots,\bar{q}_m\bar{p}_m)\in\Omega^+ S,\end{aligned}$$

so that $(q/p)[\Omega^+ K]\subset\operatorname{ord}_{\Omega^+ K} s$. Conversely, let r/t, where $r,t\in\Omega^+ K$ are coprime, be any element in $\operatorname{ord}_{\Omega^+ K} s$. Then for each $i=1,\ldots,m$,

$$\frac{r}{t}\frac{p_i}{q_i}\in\Omega^+ K.$$

Thus, q_i is a divisor of r for each i, and since q is the lcm of the q_i's, it follows that q is a divisor of r as well, that is, $r = q\bar{r}$ for some $\bar{r} \in \Omega^+ K$. Similarly, t is a divisor of each of the p_i's and hence also of p, so that $p = t\bar{p}$ for some $\bar{p} \in \Omega^+ K$. Thus,

$$\frac{r}{t} = \frac{q\bar{r}}{t} = \frac{q\bar{r}\bar{p}}{t\bar{p}} = \frac{q}{p}(\bar{r}\bar{p}),$$

so that $r/t \in (q/p)[\Omega^+ K]$, whence $\operatorname{ord}_{\Omega^+ K} s = (q/p)[\Omega^+ K]$.

(iii) ΩK *is the ring* $\Omega_{\bar{\mathfrak{D}}} K$ *of causal and stable elements.*. The quotient field $Q_{\Omega_{\bar{\mathfrak{D}}} K}$ again coincides with the usual field of rationals $Q_{\Omega^+ K}$, and an element $s \in \Lambda S$ has nonzero $\Omega_{\bar{\mathfrak{D}}} K$-order if and only if $s \in Q_{\Omega^+ K} S$. To obtain the order, let $s = (s_1, \dots, s_m) \in Q_{\Omega^+ K} S$ be a nonzero element, and write each entry s_i, $i = 1, \dots, m$, as $s_i = p_i r_i / q_i$, where $r_i, q_i \in \mathfrak{D}$ are coprime (with respect to $\Omega^+ K$), and where $(0 \neq)\ p_i \in \Omega^+ K$ is coprime with every element of $\mathfrak{D}$. Then it can be verified by direct computation that $\operatorname{ord}_{\Omega_{\bar{\mathfrak{D}}} K} s$ is generated by an element $q/rp \in Q_{\Omega^+ K}$ as follows: $p = \gcd(p_1, \dots, p_m)$, and q and r are any coprime elements of $\mathfrak{D}$ such that $\operatorname{ord}_{\Omega^- K}(q/pr) = -\operatorname{ord}_{\Omega^- K} s$.

We proceed now with our discussion of the order, starting with the following property, which can be readily verified.

PROPOSITION 3.8. *Let $s \in \Lambda S$ be an element. Then $s \in \Omega S$ if and only if $\Omega K \subset \operatorname{ord}_{\Omega K} s$.*

We consider next the behavior of the order under several operations. First we note that, for any pair $s \in Q_{\Omega K} S$ and $0 \neq \alpha \in Q_{\Omega K}$, we have $\operatorname{ord} \alpha s = \alpha^{-1}[\operatorname{ord} s]$. Hence, if $\operatorname{ord} s = \gamma[\Omega K]$, then $\operatorname{ord} \alpha s = (\alpha^{-1}\gamma)[\Omega K]$. In particular, if $\alpha \in \Omega K$, then $\operatorname{ord} s \subset \operatorname{ord} \alpha s$. Further, let $\gamma_1, \dots, \gamma_n \in Q_{\Omega K}$ be a set of elements, and let $\psi \in \Omega K$ be a nonzero element such that $\psi\gamma_i \in \Omega K$ for all $i = 1, \dots, n$. Let α be the least common multiple (in ΩK) of $\psi\gamma_1, \dots, \psi\gamma_n$. Then, we call $\gamma := \alpha/\psi$ a *least common* ΩK-*multiple* of $\gamma_1, \dots, \gamma_n$. Clearly the intersection $\gamma_1[\Omega K] \cap \cdots \cap \gamma_n[\Omega K] = \gamma[\Omega K]$. Letting $s_1, \dots, s_n \in Q_{\Omega K} S$ be a set of nonzero elements with orders $\operatorname{ord} s_i = \gamma_i[\Omega K]$, $i = 1, \dots, n$, we again have that $\operatorname{ord} s_1 \cap \cdots \cap \operatorname{ord} s_n = \gamma[\Omega K]$, where γ is the least common ΩK-multiple of $\gamma_1, \dots, \gamma_n$. Finally, considering the order of the sum $s_1 + s_2 + \cdots + s_n$, it is easy to see that

$$\operatorname{ord} s_1 \cap \cdots \cap \operatorname{ord} s_n \subset \operatorname{ord}(s_1 + \cdots + s_n). \tag{3.9}$$

A set of elements $s_1, \dots, s_n \in \Lambda S$ is ΩK-*ordered* (or, simply, *ordered*) if $\operatorname{ord} s_1 \subset \cdots \subset \operatorname{ord} s_n$.

We turn now to characterization of when a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is an ΩK-map. Recall that $\bar{f}$ is an ΩK-map if and only if $\bar{f}[\Omega U] \subset \Omega Y$, and let $0 \neq u \in Q_{\Omega K} U$ be any element. Then $\operatorname{ord} u = \gamma[\Omega K]$ for some $\gamma \in Q_{\Omega K}$ and $\gamma u \in \Omega U$. If $\bar{f}$ is an ΩK-map, then $\bar{f}(\gamma u) \in \Omega Y$, so that $\Omega K \subset \operatorname{ord} \bar{f}(\gamma u)$ (see Proposition 3.8), or, equivalently, $\Omega K \subset \operatorname{ord} \gamma \bar{f}(u) = \gamma^{-1} \operatorname{ord} \bar{f}(u)$. Thus we conclude that $\gamma[\Omega K] \subset \operatorname{ord} \bar{f}(u)$, and a necessary condition for $\bar{f}$ to be an ΩK-map is that $\operatorname{ord} u \subset \operatorname{ord} \bar{f}(u)$. This condition is actually also sufficient, and we have the following.

THEOREM 3.10. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map. Then $\bar{f}$ is an ΩK-map if and only if $\operatorname{ord} u \subset \operatorname{ord} \bar{f}(u)$ for each $u \in Q_{\Omega K} U$.*

Proof. The necessity has been seen above. The sufficiency is seen as follows. If $\bar{f}$ is not an ΩK-map, there is an element $u \in \Omega U$ satisfying the condition that $\bar{f}(u) \notin \Omega Y$. Then $\Omega K \subset \operatorname{ord} u$, but $\Omega K \not\subset \operatorname{ord} \bar{f}(u)$, so that $\operatorname{ord} u \not\subset \operatorname{ord} \bar{f}(u)$, concluding the proof. ■

The condition of Theorem 3.10 is, of course, not easily tested directly, and we would like to find a finite "test set" of elements in $Q_{\Omega K} U$ which is sufficient for verification that a ΛK-linear map is an ΩK-map. That a basis for $Q_{\Omega K} U$ may not be appropriate for this purpose is seen in the following simple example.

EXAMPLE 3.11. Let $\Omega K = \Omega^- K$, and let $Y = U = K^2$. Take as basis for $Q_{\Omega K} K^2$ the elements

$$u_1 = \begin{pmatrix} z^{-1} \\ 1 \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} z^{-2} \\ 1 \end{pmatrix},$$

and define $\bar{f}: \Lambda K^2 \to \Lambda K^2$ by

$$\bar{f}(u_1) = u_1 + u_2,$$

$$\bar{f}(u_2) = u_2.$$

Obviously, $\Omega^- K = \operatorname{ord}_{\Omega^- K} u_1 = \operatorname{ord}_{\Omega^- K} \bar{f}(u_1) = \operatorname{ord}_{\Omega^- K} u_2 = \operatorname{ord}_{\Omega^- K} \bar{f}(u_2)$. Thus, $\bar{f}$ satisfies the condition of Theorem 3.10 for the basis u_1, u_2, yet it is not an $\Omega^- K$-map (that is, not causal): since $\bar{f}(u_1 - u_2) = u_1$ and since

$$u_1 - u_2 = \begin{pmatrix} z^{-1} - z^{-2} \\ 0 \end{pmatrix},$$

we have

$$\operatorname{ord}_{\Omega^-K}(u_1-u_2)=z\Omega^-K\not\subset\operatorname{ord}_{\Omega^-K}u_1=\Omega^-K.$$

Let us explore now the cause of difficulty encountered in the above example. If $s_1,\dots,s_n\in Q_{\Omega K}S$ is a given set of elements and $\alpha_1,\dots,\alpha_n\in Q_{\Omega K}$ is any set of scalars, then by (3.9),

$$\bigcap_{i=1}^{n}\operatorname{ord}\alpha_i s_i\subset\operatorname{ord}\sum_{i=1}^{n}\alpha_i s_i.$$

But the above inclusion, in general, need not hold with equality (even when the s_i are $Q_{\Omega K}$-linearly independent). This order "deficiency" also occurs in the example, and therefore the basis selected there failed as a test set for causality. Indeed, we have there

$$\bigcap_{i=1}^{2}\operatorname{ord}_{\Omega^-K}u_i=\Omega^-K\neq\operatorname{ord}_{\Omega^-K}(u_1-u_2)=z\Omega^-K.$$

Thus, we are motivated to introduce the following

DEFINITION 3.12. A set of nonzero elements $s_1,\dots,s_n\in Q_{\Omega K}S$ is called *ΩK-adapted* if for every set of scalars $\alpha_1,\dots,\alpha_n\in Q_{\Omega K}$ the condition

$$\bigcap_{i=1}^{n}\operatorname{ord}\alpha_i s_i=\operatorname{ord}\sum_{i=1}^{n}\alpha_i s_i \tag{3.13}$$

holds. A basis of ΩK-adapted elements $s_1,\dots,s_m$ of $Q_{\Omega K}S$ is called an *ΩK-adapted* basis.

It is easily verified that in Definition 3.12 we could replace $Q_{\Omega K}$ by ΩK, i.e., $s_1,\dots,s_n$ *is ΩK-adapted if and only if* (3.13) *holds for every set* $\alpha_1,\dots,\alpha_n\in\Omega K$.

It is important to note that Definition 3.12 reduces to Proposition 3.2 in the particular case when ΩK is the ring of power series Ω^-K [see Example 3.7(i)]. Thus, Definition 3.12 forms a natural generalization of the notion of proper bases to the case of general principal ideal domains, in a framework in which the classical notion of order (3.1) is replaced by $\operatorname{ord}_{\Omega K}$. We start our investigation of adapted sets with the following

THEOREM 3.14. *An ΩK-adapted set of nonzero elements $s_1,\dots,s_n\in Q_{\Omega K}S$ is ΛK-linearly independent.*

Proof. Assume the set $s_1,\ldots,s_n \in Q_{\Omega K}S$ is ΛK-linearly dependent. Then, since $s_1,\ldots,s_n$ are ΩK-rational, they are also $Q_{\Omega K}$-linearly dependent, and there are elements $\alpha_1,\ldots,\alpha_n \in Q_{\Omega K}$, not all zero, such that $\sum_{i=1}^n \alpha_i s_i = 0$. If the set is ΩK-adapted, then (3.13) holds, and we have that

$$\bigcap_{i=1}^{n} \operatorname{ord} \alpha_i s_i = \operatorname{ord} 0 = Q_{\Omega K}.$$

Thus, it follows that $\operatorname{ord} \alpha_i s_i = Q_{\Omega K}$, $i = 1,\ldots,n$, implying that $\alpha_i s_i = 0$ for all $i = 1,\ldots,n$, a contradiction, since we assumed that all the s_i's are nonzero. ■

Let $s_1,\ldots,s_n \in \Lambda S$ be a set of elements, and let $\Lambda[s_1,\ldots,s_n]$ denote the ΛK-linear space spanned by $s_1,\ldots,s_n$. We then have the following characterization of ΩK-adapted sets.

THEOREM 3.15. *Consider a set of nonzero elements $s_1,\ldots,s_n \in Q_{\Omega K}S$ with $\operatorname{ord} s_i = \gamma_i[\Omega K]$, $i = 1,\ldots,n$. Then $(s_1,\ldots,s_n)$ is an ΩK-adapted set if and only if $(\gamma_1 s_1,\ldots,\gamma_n s_n)$ forms a basis for the ΩK-module $\Lambda[s_1,\ldots,s_n] \cap \Omega S$.*

Proof. "Only if": First note that from the definition of order, the ΩK-module $\Delta_0 := \Omega[\gamma_1 s_1,\ldots,\gamma_n s_n]$, generated by $(\gamma_1 s_1,\ldots,\gamma_n s_n)$, is contained in $\Delta := \Lambda[s_1,\ldots,s_n] \cap \Omega S$. To see that the converse inclusion $\Delta \subset \Delta_0$ also holds, let $s = \sum_{i=1}^n \alpha_i s_i$ $(\in \Omega S)$ be any element of Δ. If $s_1,\ldots,s_n$ is an ΩK-adapted set, then, by (3.13), $\operatorname{ord} s = \bigcap_{i=1}^n \operatorname{ord} \alpha_i s_i$, and by Proposition 3.8 $\Omega K \subset \operatorname{ord} s \subset \operatorname{ord} \alpha_i s_i$, $i = 1,\ldots,n$. Thus, there are elements $\beta_i \in \Omega K$, $i = 1,\ldots,n$, such that $\alpha_i = \beta_i \gamma_i$ and we have $s = \sum_{i=1}^n \beta_i \gamma_i s_i \in \Delta_0$ as claimed.

"If": Assume that the set $(\gamma_1 s_1,\ldots,\gamma_n s_n)$ forms a basis for Δ, and consider any element $s = \sum_{i=1}^n \alpha_i s_i$ where $\alpha_1,\ldots,\alpha_n \in Q_{\Omega K}$ are not all zero. The proof will be complete upon showing that $\operatorname{ord} s = \bigcap_{i=1}^n \operatorname{ord} \alpha_i s_i$, and since the inclusion $\bigcap_{i=1}^n \operatorname{ord} \alpha_i s_i \subset \operatorname{ord} s$ is obvious, it remains only to show that the converse inclusion holds. To prove the latter, let $\operatorname{ord} s = \gamma[\Omega K]$. Then $\gamma s \in \Delta$ and, by assumption, there are elements $\beta_1,\ldots,\beta_n \in \Omega K$ such that γs $(= \sum_{i=1}^n \gamma \alpha_i s_i) = \sum_{i=1}^n \beta_i \gamma_i s_i$. By the uniqueness of the representation it follows that $\gamma \alpha_i = \beta_i \gamma_i$, $i = 1,\ldots,n$, and we have that $\gamma \alpha_i s_i = \beta_i(\gamma_i s_i) \in \Omega S$ for $i = 1,\ldots,n$. Hence $\gamma \in \bigcap_{i=1}^n \operatorname{ord} \alpha_i s_i$, concluding the proof. ■

From the above theorem we directly obtain the following characterization of ΩK-adapted bases.

COROLLARY 3.16. *Assume the set $s_1,\ldots,s_m \in Q_{\Omega K}S$ is a basis for ΛS with $\operatorname{ord} s_i = \gamma_i[\Omega K]$, $i = 1,\ldots,m$. Then the set $(s_1,\ldots,s_m)$ is ΩK-adapted if and only if $(\gamma_1 s_1,\ldots,\gamma_m s_m)$ generates ΩS.*

One of the fundamental properties of order preserving maps is that they transform adapted sets into adapted sets, as follows.

PROPOSITION 3.20. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map, and let $u_1, \ldots, u_n \in Q_{\Omega K} U$ be an ΩK-adapted set. If $\bar{f}$ is order preserving, then $\bar{f}(u_1), \ldots, \bar{f}(u_n) \in \Lambda Y$ is also an ΩK-adapted set.*

Proof. We need to show that for every set $\alpha_1, \ldots, \alpha_n \in Q_{\Omega K}$, $\bigcap_{i=1}^n \operatorname{ord} \alpha_i \bar{f}(u_i) = \operatorname{ord} \sum_{i=1}^n \alpha_i \bar{f}(u_i)$. Indeed,

$$\bigcap_{i=1}^{n} \operatorname{ord} \alpha_i \bar{f}(u_i) = \bigcap_{i=1}^{n} \alpha_i^{-1} \operatorname{ord} \bar{f}(u_i)$$

by the order preserving property

$$= \bigcap_{i=1}^{n} \alpha_i^{-1} \operatorname{ord} u_i$$

$$= \bigcap_{i=1}^{n} \operatorname{ord} \alpha_i u_i$$

since the u_i's are ΩK-adapted

$$= \operatorname{ord} \sum_{i=1}^{n} \alpha_i u_i$$

by the order preserving property

$$= \operatorname{ord} \bar{f}\left(\sum_{i=1}^{n} \alpha_i u_i \right)$$

$$= \operatorname{ord} \sum_{i=1}^{n} \alpha_i \bar{f}(u_i). \qquad \blacksquare$$

We can now state a full characterization of order preserving maps.

THEOREM 3.21. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map, and let $u_1, \ldots, u_m \in Q_{\Omega K} U$ be an ΩK-adapted basis for ΛU. Then $\bar{f}$ is ΩK-order preserving if and only if* (i) *$\bar{f} u_1, \ldots, \bar{f} u_m$ are ΩK-adapted, and* (ii) $\operatorname{ord} u_i = \operatorname{ord} \bar{f} u_i$ *for all $i = 1, \ldots, m$.*

Example 3.17. Corollary 3.16 provides a particularly simple way to determine whether a basis $s_1,\dots,s_m$ of the ΛK-linear space ΛS is ΩK-adapted. Indeed, $(\gamma_1 s_1,\dots,\gamma_m s_m)$ generate ΩS if and only if the matrix $[\gamma_1 s_1,\dots,\gamma_m s_m]$ is ΩK-unimodular. Thus, the main clause of the corollary can be restated to read: *The basis $s_1,\dots,s_m$ of ΛS is ΩK-adapted if and only if* $\det[s_1,\dots,s_m] = \gamma_1^{-1}\gamma_2^{-1}\cdots\gamma_m^{-1}\mu$, *where μ is a unit in ΩK.* As an illustration of this simple criterion, we show that the columns

$$s_1 = \begin{pmatrix} z \\ z^3 \\ z^4 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} z^2+1 \\ (z^2+1)^2 \\ z^4(z^2+1) \end{pmatrix}, \qquad s_3 = \begin{pmatrix} 0 \\ 0 \\ z^3+1 \end{pmatrix}$$

form an (unordered) Ω^+K-adapted basis for ΛK^3. Indeed, we have $\operatorname{ord}_{\Omega^+K} s_1 = (z^{-1})[\Omega^+K]$, $\operatorname{ord}_{\Omega^+K} s_2 = ((z^2+1)^{-1})[\Omega^+K]$, and $\operatorname{ord}_{\Omega^+K} s_3 = ((z^3+1)^{-1})[\Omega^+K]$, whence $\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1} = z(z^2+1)(z^3+1)$, which is equal to $\det[s_1, s_2, s_3]$. If however, s_1 (say) is replaced by $s_1' = (2z, z^3, z^4)^T$, then the resulting set will no longer be Ω^+K-adapted, since $\det[s_1', s_2, s_3] = (z^3+1)(z^2+1)(z^3+2z)$.

We arrive now at a finitary characterization of ΩK-maps, which is in complete analogy to Theorem 3.3.

Theorem 3.18. *Let $\bar{f}\colon \Lambda U \to \Lambda Y$ be a ΛK-linear map, and assume that $u_1,\dots,u_m$ is an ΩK-adapted basis for ΛU. Then $\bar{f}$ is an ΩK-map if and only if* $\operatorname{ord} u_i \subset \operatorname{ord} \bar{f}(u_i)$ *for all $i = 1,\dots,m$.*

Proof. By Theorem 3.10 the condition is clearly necessary. To see sufficiency, assume that $\operatorname{ord} u_i = \gamma_i[\Omega K]$, $i = 1,\dots,m$. By Corollary 3.16 the set $(\gamma_1 u_1,\dots,\gamma_m u_m)$ generates ΩU. But, since $\operatorname{ord} u_i \subset \operatorname{ord} \bar{f}(u_i)$ for all $i = 1,\dots,m$, it follows that $\gamma_i \bar{f}(u_i) = \bar{f}(\gamma_i u_i) \in \Omega Y$, whence $\bar{f}[\Omega U] \subset \Omega Y$ and $\bar{f}$ is an ΩK-map. ■

Consider an ΩK-unimodular map $\bar{l}\colon \Lambda U \to \Lambda U$. Clearly, for every pair $u, v \in \Lambda U$, we have (see Theorem 3.10) that $\operatorname{ord} u \subset \operatorname{ord} \bar{l}u$ and $\operatorname{ord} v \subset \operatorname{ord} \bar{l}^{-1}v$. Substituting $v = \bar{l}u$, we obtain that also $\operatorname{ord} \bar{l}u \subset \operatorname{ord} u$, so that $\operatorname{ord} \bar{l}u = \operatorname{ord} u$ for every $u \in \Lambda U$. Thus, an ΩK-unimodular map preserves the ΩK-order. We now generalize this notion.

Definition 3.19. A ΛK-linear map $\bar{f}\colon \Lambda U \to \Lambda Y$ is called *ΩK-order preserving* (or, simply, *order preserving*) if for each $u \in Q_{\Omega K}U$, $\operatorname{ord} u = \operatorname{ord} \bar{f}(u)$.

Proof. The necessity of conditions (i) and (ii) follows directly from Proposition 3.20 and Definition 3.19, respectively. We now prove the "if" direction. Assume that conditions (i) and (ii) hold, and let $u = \sum_{i=1}^{m} \alpha_i u_i$, where $\alpha_1, \ldots, \alpha_m \in Q_{\Omega K}$, be any element in $Q_{\Omega K}U$. Then since the u_i's are ΩK-adapted, we have by (3.13) that $\operatorname{ord} u$ ($= \operatorname{ord} \sum_{i=1}^{m} \alpha_i u_i$) $= \bigcap_{i=1}^{m} \operatorname{ord} \alpha_i u_i$, and it follows that

$$
\begin{aligned}
\operatorname{ord} u &= \bigcap_{i=1}^{m} \operatorname{ord} \alpha_i u_i \\
&= \bigcap_{i=1}^{m} \operatorname{ord} \alpha_i \bar{f} u_i \qquad [\text{since } \operatorname{ord} \bar{f} u_i = \operatorname{ord} u_i] \\
&= \operatorname{ord}\left(\sum_{i=1}^{m} \alpha_i \bar{f} u_i \right) \qquad [\text{by (i)}] \\
&= \operatorname{ord} \bar{f}\left(\sum_{i=1}^{m} \alpha_i u_i \right) \\
&= \operatorname{ord} \bar{f}(u),
\end{aligned}
$$

whence $\bar{f}$ is order preserving. ∎

We can now prove the converse direction of our previous observation that ΩK-unimodular maps are order preserving.

COROLLARY 3.22. *Let $\bar{f}: \Lambda U \to \Lambda U$ be a ΛK-linear map. Then $\bar{f}$ is ΩK-unimodular if and only if it is ΩK-order preserving.*

Proof. The "only if" direction was considered above. Conversely, if $\bar{f}$ is order preserving then it is clearly injective, and hence is a ΛK-linear isomorphism $\Lambda U \cong \Lambda U$, so that $\bar{f}^{-1}$ exists. By assumption, $\operatorname{ord} \bar{f} u = \operatorname{ord} u$ for all $u \in \Lambda U$. Letting $v := \bar{f} u$, we obtain $\operatorname{ord} v = \operatorname{ord} \bar{f}^{-1} v$ for all $v \in \Lambda U$. Thus, by Theorem 3.10, both of $\bar{f}$ and $\bar{f}^{-1}$ are ΩK-maps, and $\bar{f}$ is ΩK-unimodular. ∎

4. BOUNDED ΩK-MODULES AND THE EXISTENCE OF ADAPTED BASES

Before considering the existence of ΩK-adapted bases, it is helpful to study a particular type of ΩK-submodules of ΛS. Let $\Delta \subset \Lambda S$ be an ΩK-module. We say that Δ is *ΩK-rational* if it consists exclusively of ΩK-rational elements. An ΩK-module $\Delta \subset \Lambda S$ is *ΩK-bounded* if there exists a nonzero element $\gamma \in Q_{\Omega K}$ such that $\gamma[\Delta] \subset \Omega S$ (i.e., $\gamma s \in \Omega S$ for every $s \in \Delta$). Let $\Delta \subset \Lambda S$ be a bounded ΩK-module. We define the *order* of Δ, denoted $\mathrm{ord}_{\Omega K} \Delta$, as the class of all elements $\gamma \in Q_{\Omega K}$ satisfying $\gamma[\Delta] \subset \Omega S$. It is easily seen that $\mathrm{ord}_{\Omega K} \Delta = \bigcap_{s \in \Delta} \mathrm{ord}_{\Omega K} s$. If Δ is a nonzero submodule and $0 \neq s \in \Delta$ is any element, then $\mathrm{ord}\, \Delta \subset \mathrm{ord}\, s$, so that from the fact that ΩK is a principal ideal domain and $\mathrm{ord}\, s$ is a cyclic module (Proposition 3.6) it follows that also $\mathrm{ord}\, \Delta$ is cyclic and $\mathrm{rank}_{\Omega K} \Delta = 1$. Thus, if $\Delta \neq 0$ there is an element $\psi \in Q_{\Omega K}$ such that $\mathrm{ord}_{\Omega K} \Delta = \psi[\Omega K]$. Otherwise, if $\Delta = 0$, we have the $\mathrm{ord}_{\Omega K} \Delta = Q_{\Omega K}$.

Clearly, every bounded ΩK-module is necessarily ΩK-rational as well. The converse, however, is not true in general, and a rational ΩK-module may be not bounded. For example when ΩK is the ring of power series, then the space ΛS is a rational ΩK-module, but it is evidently not bounded. Nevertheless, the following is true.

LEMMA 4.1. *Let $\Delta \subset \Lambda S$ be a rational ΩK-submodule. Then Δ is bounded if and only if Δ has finite rank (i.e., is finitely generated), in which case* $\mathrm{rank}\, \Delta \leqslant \dim S$.

Proof. "Only if": Let Δ be bounded, and let $\mathrm{ord}\, \Delta = \psi[\Omega K]$ with $0 \neq \psi \in Q_{\Omega K}$. Then $\psi\Delta \subset \Omega S$, so that, in view of the fact that ΩK is a principal ideal domain, $\mathrm{rank}\, \psi\Delta \leqslant \mathrm{rank}\, \Omega S = \dim S$. But, clearly, $\mathrm{rank}\, \Delta = \mathrm{rank}\, \psi\Delta$, concluding the proof of the "only if" part.

"If": Assume Δ has finite rank, and let $d_1, \dots, d_n \in \Delta$ with $\mathrm{ord}\, d_i = (\gamma_i)_{\Omega K}$ be a basis. Then, since (d_i) are rational, $\gamma_i \neq 0$ for all $i = 1, \dots, n$, and, by definition, for every $d \in \Delta$ there are elements $\alpha_1, \dots, \alpha_n \in \Omega K$ such that $d = \sum_{i=1}^n \alpha_i d_i$. But then

$$\mathrm{ord}\, d = \mathrm{ord} \sum_{i=1}^{n} \alpha_i d_i$$

$$\supset \bigcap_{i=1}^{n} \mathrm{ord}\, \alpha_i d_i$$

[since, for every $\alpha_i \in \Omega K$, $\mathrm{ord}\, d_i = \alpha_i[\mathrm{ord}\, \alpha_i d_i] \subset \mathrm{ord}\, \alpha_i d_i$, $i = 1, \dots, n$],

$$\supset \bigcap_{i=1}^{n} \mathrm{ord}\, d_i =: \psi[\Omega K]$$

where ψ is the least common ΩK-multiple of $\gamma_1, \dots, \gamma_n$.

Further, since $\gamma_i \neq 0$ for all $i = 1,\ldots,n$, we have that $\psi \neq 0$, so that, since by construction $\psi[\Delta] \subset \Omega S$, the module Δ is bounded. That $\operatorname{rank} \Delta \leq \dim S$ follows from $\operatorname{rank} \Delta = \operatorname{rank} \psi[\Delta]$, since $\psi[\Delta] \subset \Omega S$. ■

In [5, Theorem 6.11] it was shown (in our present terminology) that every bounded $\Omega^- K$-module has an $\Omega^- K$-adapted basis. Actually, this result is just a manifestation of the following general statement.

THEOREM 4.2. *Let $\Delta \subset \Lambda S$ be a nonzero bounded ΩK-module. Then:*

(i) *Δ has an ordered ΩK-adapted basis $d_1,\ldots,d_r$.*

(ii) *If $d_1^1,\ldots,d_r^1$ is any other ordered ΩK-adapted basis of Δ, then* $\operatorname{ord} d_i^1 = \operatorname{ord} d_i$, $i = 1,\ldots,r$.

Before proving Theorem 4.2, it will be convenient to recall the Smith canonical form theorem (see, e.g., [12]).

THEOREM 4.3. *Let T be an $m \times n$ ΩK-matrix. Then there are ΩK-unimodular matrices M_L and M_R of dimensions $m \times m$ and $n \times n$, respectively, and elements $\delta_1,\ldots,\delta_r \in \Omega K$, uniquely defined up to multiples of units of ΩK, where $r \leq \min(m,n)$ and δ_{i+1} divides δ_i for all $i = 1,\ldots,r-1$, such that*

$$T = M_L D M_R, \tag{4.4}$$

where D is the $m \times n$ matrix given by $D = \operatorname{diag}(\delta_1,\ldots,\delta_r,0,\ldots,0)$.

The elements $\delta_1,\ldots,\delta_r$ in Theorem 4.3 are called the *invariant factors* of T.

Proof of Theorem 4.2: Assume that $\Delta \subset \Lambda S$ with $\dim S = m$ is a bounded ΩK-module with $\operatorname{ord} \Delta = \psi[\Omega K]$, and, in view of Lemma 4.1, let $d_1,\ldots,d_r \in \Delta$ be a basis for Δ. Then $\psi d_1,\ldots,\psi d_r \in \Omega S$, and the $m \times r$ matrix $\psi T := [\psi d_1,\ldots,\psi d_r]$ (where ψd_i is viewed as a column vector) has Smith representation

$$\psi T = M_L D M_R, \tag{4.5}$$

where

$$D = \left[\begin{array}{ccc} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_r \\ \hline & 0 & \end{array}\right].$$

and the $\delta_i \in \Omega K$ (with δ_{i+1} dividing δ_i) are the invariant factors of ψT. We note that, by definition of r, $\delta_i \neq 0$ for all $i = 1,\dots,r$. Dividing both sides of (4.5) by ψ yields

(4.6) $$T = M_L D_0 M_R$$

where D_0 is the *Smith-McMillan form* of D and is given by

$$D_0 = \left[\begin{array}{ccc} \delta_1/\psi & & 0 \\ & \ddots & \\ 0 & & \delta_r/\psi \\ \hline & 0 & \end{array}\right]$$

Let d_{0i} denote the ith column of D_0. The columns $d_{01},\dots,d_{0r} \in Q_{\Omega K}S$ constitute an ΩK-adapted set, since for every set $\alpha_1,\dots,\alpha_r \in Q_{\Omega K}$ we have that

$$d := \sum_{i=1}^{r} \alpha_i d_{0i} = \begin{bmatrix} \alpha_1 \dfrac{\delta_1}{\psi} \\ \vdots \\ \alpha_r \dfrac{\delta_r}{\psi} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and clearly $\operatorname{ord} d = \cap_{i=1}^{r} \operatorname{ord}(\alpha_i \delta_i/\psi) = \cap_{i=1}^{r} \operatorname{ord} \alpha_i d_{0i}$. Furthermore, since M_L is ΩK-unimodular, it follows by Proposition 3.20 that the columns of $M_L D_0$, given by $(\delta_1/\psi)M_{L1},\dots,(\delta_r/\psi)M_{Lr}$ (where M_{Li} is the ith column of M_L), are ΩK-adapted as well.

Now, since M_R is ΩK-unimodular, we have that $\Delta = T[\Omega S] = M_L D_0 M_R[\Omega S] = M_L D_0[\Omega S]$, so that the columns of $M_L D_0$ form a basis of Δ, and, as we have just shown, this basis is ΩK-adapted. To show that this basis is also ordered, we note that, since the greatest common ΩK-divisor of all entries in M_{Li} is 1 for all $i = 1,\dots,r$, we have $\operatorname{ord}(\delta_i/\psi)M_{Li} = (\psi/\delta_i)[\Omega K]$, $i = 1,\dots,r$. Hence, since δ_{i+1} divides δ_i for all $i = 1,\dots,r-1$, we obtain $\operatorname{ord}(\delta_r/\psi)M_{Lr} \subset \cdots \subset \operatorname{ord}(\delta_1/\psi)M_{L1}$. Thus, the columns $(\delta_r/\psi)M_{Lr}$, $(\delta_{r-1}/\psi)M_{L,r-1},\dots,(\delta_1/\psi)M_{L1}$ form an ordered adapted basis of Δ. This concludes our proof. ∎

Let $\Delta \subset \Lambda S$ be a bounded ΩK-module, and let $d_1, \ldots, d_r$ be an ordered adapted basis of Δ. We call the set $\operatorname{ord} d_1 \subset \operatorname{ord} d_2 \subset \cdots \subset \operatorname{ord} d_r$ the *order trace* of Δ. In view of Theorem 4.2(ii), the order trace is uniquely determined by Δ. It is also easy to see that $\operatorname{ord} \Delta = \operatorname{ord} d_1$. Letting $D := [d_1, \ldots, d_r]$ be the corresponding matrix, we can represent Δ as $\Delta = D[\Omega K^r]$. In case $\operatorname{rank} \Delta = \dim_K S$, we say that the module Δ is *full*.

THEOREM 4.7. *Let $\Delta_1, \Delta_2 \subset \Lambda S$ be bounded ΩK-submodules given by $\Delta_1 = D_1 \Omega S$ and $\Delta_2 = \Delta_2 \Omega S$, respectively. Then $\Delta_2 \subset \Delta_1$ if and only if there exists an ΩK-matrix R (i.e., with entries in ΩK) such that $D_2 = D_1 R$.*

Proof. Elementary. ■

COROLLARY 4.8. *Let $\Delta_1, \Delta_2 \subset \Lambda S$ be bounded ΩK-submodules given by $\Delta_1 = D_1 \Omega S$ and $\Delta_2 = D_2 \Omega S$. Assume Δ_1 is full, and define $R := D_1^{-1} D_2$. Then $\Delta_2 \subseteq \Delta_1$ if and only if R is an ΩK-matrix, with equality holding if and only if R is ΩK-unimodular.*

We turn now to the existence of ΩK-adapted bases for ΛK-linear spaces. A ΛK-linear subspace $\mathfrak{R} \subset \Lambda S$ is called *ΩK-rational* if it has a basis $s_1, \ldots, s_k$ consisting of ΩK-rational vectors.

THEOREM 4.9. *Let $\dim S = m$, and let $\mathfrak{R} \subset \Lambda S$ be a nonzero ΩK-rational ΛK-linear subspace. Then* (i) *$\mathfrak{R}$ has an ΩK-adapted basis, and* (ii) *every ΩK-adapted subset $s_1, \ldots, s_l \in \mathfrak{R}$ can be extended to an ΩK-adapted basis for $\mathfrak{R}$.*

Proof. (i): Let $s_1, \ldots, s_k$ be an ΩK-rational basis for $\mathfrak{R}$, and write $R = [s_1, \ldots, s_k]$ (where the s_i's are regarded as column vectors). The $m \times k$ ΩK-rational matrix R has a Smith-McMillan representation

$$R = M_L D M_R,$$

where M_L and M_R are ΩK-unimodular, and where D is the Smith-McMillan form of R. Then $\mathfrak{R} = R[\Lambda K^k] = M_L D M_R[\Lambda K^k] = M_L D[\Lambda K^k]$, and the columns of $M_L D$ constitute an ΩK-adapted basis for $\mathfrak{R}$ (see proof of Theorem 4.2).

(ii): Let $s_1, \ldots, s_l \in \mathfrak{R}$ constitute an ΩK-adapted set. We shall demonstrate a procedure for extending this set to an ΩK-adapted basis for $\mathfrak{R}$. First recall that the set $s_1, \ldots, s_l$ is ΛK-linearly independent (Theorem 3.14), and hence can be extended to an ΩK-rational basis $s_1, \ldots, s_k$ of $\mathfrak{R}$. Let $s_{l+1}, \ldots, s_k$ be such an extension. Define the matrices $R: = [s_1, \ldots, s_l]$ and $\hat{R} = [s_{l+1}, \ldots, s_k]$. Now, let M_L be an ΩK-unimodular matrix such that

$$R = M_L D,$$

where

$$D = \begin{pmatrix} D_0 \\ 0 \end{pmatrix}$$

and D_0 is a square $(l \times l)$ matrix (the existence of M_L follows by the Hermite normal form theorem; see e.g. [12]). By Proposition 3.20, the columns of D_0 are still ΩK-adapted. Next, decompose the representation as

$$R = [M_L^1, M_L^2]\begin{bmatrix} D_0 \\ 0 \end{bmatrix} = M_L^1 D_0,$$

where M_L^1 is $m \times l$. Let $\gamma \in Q_{\Omega K}$ be a nonzero element such that γD_0^{-1} is an ΩK-matrix, and let $0 \neq \psi \in \bigcap_{i=l+1}^{k} \operatorname{ord} s_i$ be any element, so that $\psi \hat{R}$ is an ΩK-matrix. Define the matrix $\bar{R}: = \gamma\psi\hat{R}$. Clearly, the columns of $[R, \bar{R}]$ still form a basis for $\mathfrak{R}$. Now, upon defining $R^1 = M_L^{-1}\bar{R}$, we obtain

$$(4.10) \qquad [R, \bar{R}] = M_L[D, R^1] = [M_L^1, M_L^2]\begin{bmatrix} D_0 & R_1^1 \\ 0 & R_2^1 \end{bmatrix},$$

where

$$R^1 = \begin{bmatrix} R_1^1 \\ R_2^1 \end{bmatrix}$$

is a decomposition of R^1 such that R_2^1 is $(m-l)\times(k-l)$. In view of the nonsingularity of D_0 and the fact that $\bar{R} = \gamma\psi\hat{R}$ and $\psi\hat{R}$ is an ΩK-matrix, it follows that the matrix $P: = D_0^{-1}R_1^1$ is an ΩK-matrix. Now we can write

$$[R, \bar{R}] = [M_L^1, M_L^2]\begin{bmatrix} D_0 & 0 \\ 0 & R_2^1 \end{bmatrix}\begin{bmatrix} I & P \\ 0 & I \end{bmatrix}.$$

Further, let

$$R_2^1 = \hat{M}_L \hat{D} \hat{M}_R$$

be the Smith-McMillan representation of R_2^1. Continuing from (4.10), we then have

$$\begin{aligned}
[R, \bar{R}] &= [M_L^1, M_L^2]\begin{bmatrix} D_0 & 0 \\ 0 & R_2^1 \end{bmatrix}\begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \\
&= [M_L^1, M_L^2]\begin{bmatrix} D_0 & 0 \\ 0 & \hat{M}_L \hat{D} \hat{M}_R \end{bmatrix}\begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \\
&= [M_L^1, M_L^2 \hat{M}_L]\begin{bmatrix} D_0 & 0 \\ 0 & \hat{D} \end{bmatrix}\begin{bmatrix} I & P \\ 0 & \hat{M}_R \end{bmatrix}.
\end{aligned}$$

Now, the matrix $\begin{bmatrix} I & P \\ 0 & \hat{M}_R \end{bmatrix}$ is clearly ΩK-unimodular, so that the columns of the matrix

$$D: = [M_L^1, M_L^2 \hat{M}_L]\begin{bmatrix} D_0 & 0 \\ 0 & \hat{D} \end{bmatrix}$$

also span $\mathfrak{R}$. Moreover, we claim that the columns of D form an ΩK-adapted set. Indeed, by construction, the columns of D_0 form an ΩK-adapted set, and, since $\hat{D}$ is diagonal, its columns also form an ΩK-adapted set. This implies that the columns of the block diagonal matrix $\begin{bmatrix} D_0 & 0 \\ 0 & \hat{D} \end{bmatrix}$ form an ΩK-adapted set. But then, since the matrix $[M_L^1, M_L^2 \hat{M}_L]$ is ΩK-unimodular, it follows by Proposition 3.20 that the columns of D form an ΩK-adapted basis of Δ. Finally, noting that $D = [R, M_L^2 \hat{M}_L \hat{D}]$, we obtain that the columns of $M_L^2 \hat{M}_L \hat{D}$ extend $s_1, \ldots, s_l$ into an ΩK-adapted basis of $\mathfrak{R}$, concluding our proof. ■

We are now in a position to give an algebraic characterization of the order trace.

PROPOSITION 4.11. *Let $\Delta, \Delta^1 \subset \Lambda S$ be nonzero and bounded ΩK-modules of equal rank n. Then there exists an ΩK-unimodular map $M: \Lambda S \to \Lambda S$ such that $M[\Delta] = \Delta^1$ if and only if Δ and Δ^1 have the same order traces.*

Proof. Assume first that an ΩK-unimodular map M exists such that $M[\Delta]=\Delta^1$, and let $d_1,\dots,d_n$ be an ordered ΩK-adapted basis for Δ. Clearly then, the set $d_1^1:=Md_1,\dots,d_n^1:=Md_n$ is a basis for Δ^1. Moreover, since M is order preserving, it follows by Proposition 3.20 that this basis is in fact ordered and ΩK-adapted. Thus, in view of Theorem 4.2, the order traces of Δ and Δ^1 are the same.

Conversely, let Δ and Δ^1 have the same order traces, and let $d_1,\dots,d_n$ and $d_1^1,\dots,d_n^1$ be ordered ΩK-adapted bases for Δ and Δ^1, respectively. Extend, as in Theorem 4.9, these bases of Δ and Δ^1 to ΩK-adapted bases for ΛS: $d_1,\dots,d_n,\ d_{n+1},\dots,d_m$ and $d_1^1,\dots,d_n^1,\ d_{n+1}^1,\dots,d_m^1$, respectively. Let γ_i and γ_i^1, respectively, be generators of $\operatorname{ord} d_i$ and $\operatorname{ord} d_i^1$, $i=n+1,\dots,m$, and define the ΛK-linear map $M:\Lambda S\to\Lambda S$ through

$$Md_i=\begin{cases} d_i^1, & i=1,\dots,n,\\ \gamma_i^{-1}\gamma_i^1 d_i^1, & i=n+1,\dots,m.\end{cases}$$

Clearly $\operatorname{ord} Md_i=\operatorname{ord} d_i$ for all $i=1,\dots,m$, and, since both of the bases are adapted, Theorem 3.21 and Corollary 3.22 imply that M is ΩK-unimodular. That $M[\Delta]=\Delta^1$ follows from the construction. ∎

Related to the notion of ΩK-adapted bases is also the following

DEFINITION 4.12. Let $\mathfrak{R}_1,\dots,\mathfrak{R}_k\subset\Lambda S$ be ΩK-rational ΛK-linear subspaces. Then $\mathfrak{R}_1,\dots,\mathfrak{R}_k$ are called *ΩK-adapted* if for every set of elements $s_1,\dots,s_k$, where $s_i\in\mathfrak{R}_i$, $i=1,\dots,k$,

$$\operatorname{ord}(s_1+\cdots+s_k)=\bigcap_{i=1}^{k}\operatorname{ord} s_i.$$

It follows readily from the above definition that the concept of ΩK-adapted subspaces is equivalent to the following: Let $\mathfrak{R}_1,\dots,\mathfrak{R}_k\in\Lambda S$ be ΩK-rational ΛK-linear subspaces; and let $d_{i1},\dots,d_{il_i}$ be a basis for $\mathfrak{R}_i$, $i=1,\dots,k$. Then the subspaces $\mathfrak{R}_1,\dots,\mathfrak{R}_k$ are ΩK-adapted if and only if $d_{11},\dots,d_{1l_1},\dots,d_{k1},\dots,d_{kl_k}$ is an ΩK-adapted basis for $\mathfrak{R}_1+\cdots+\mathfrak{R}_k$. Naturally, ΩK-adapted spaces are ΛK-linearly independent so that the above sum of subspaces is, in fact, a direct sum. Accordingly, we speak of *ΩK-adapted direct sums* of ΛK-linear spaces.

The concept of ΩK-adapted subspaces is of course a generalization to arbitrary P.I.D.'s of the concept of properly independent and stably independent spaces as defined in [5,7,8].

Theorem 4.9 leads to the following useful result.

COROLLARY 4.13. *Let* $\mathfrak{R}_1 \subset \mathfrak{R}_2$ $(\subset \Lambda S)$ *be* ΩK*-rational* ΛK*-linear subspaces. Then* $\mathfrak{R}_1$ *has an* ΩK*-adapted direct summand in* $\mathfrak{R}_2$.

5. ΩK-FACTORIZATION AND INVERTIBILITY

In the present section we consider the following factorization problem. Let $\bar{f}_1 : \Lambda U \to \Lambda Y$ and $\bar{f}_2 : \Lambda U \to \Lambda W$ be ΛK-linear maps, and let $\Omega K \subsetneq \Lambda K$ be a principal ideal domain. Under what conditions does there exist an ΩK-map $\bar{h} : \Lambda Y \to \Lambda W$ such that $\bar{f}_2 = \bar{h}\bar{f}_1$? We first give an abstract version of the factorization conditions, and then we state them in explicit matrix form. Assume first that there exists an ΩK-map $h : \Lambda Y \to \Lambda W$ such that $\bar{f}_2 = \bar{h} \cdot \bar{f}_1$, and choose any element $u \in \Lambda U$ which satisfies the condition $\bar{f}_1(u) \in \Omega Y$, or, in the notation of (2.10), that $u \in \ker \pi_{\Omega K} \bar{f}_1$. Then, obviously, $\bar{f}_2(u) = \bar{h} \cdot \bar{f}_1(u) \in \Omega W$, so that $u \in \ker \pi_{\Omega K} \bar{f}_2$. Thus, the existence of the ΩK-map $\bar{h}$ satisfying $\bar{f}_2 = \bar{h}\bar{f}_1$ implies that $\ker \pi_{\Omega K} \bar{f}_1 \subset \ker \pi_{\Omega K} \bar{f}_2$. In case the maps $\bar{f}_1$ and $\bar{f}_2$ are ΩK-rational, the converse of this statement is also true, and we have the following

THEOREM 5.1. *Let* $\bar{f}_1 : \Lambda U \to \Lambda Y$ *and* $\bar{f}_2 : \Lambda U \to \Lambda W$ *be* ΩK*-rational* ΛK*-linear maps. There exists an* ΩK*-map* $\bar{h} : \Lambda Y \to \Lambda W$ *such that* $\bar{f}_2 = \bar{h} \cdot \bar{f}_1$ *if and only if* $\ker \pi_{\Omega K} \bar{f}_1 \subset \ker \pi_{\Omega K} \bar{f}_2$.

We prove Theorem 5.1 with the aid of the following lemmas.

LEMMA 5.2. *Let* $\bar{f} : \Lambda U \to \Lambda Y$ *be an* ΩK*-rational* ΛK*-linear map. Let* $r := \dim_{\Lambda K} \operatorname{Im} \bar{f}$, *and let* $Y_0 \subset Y$ *be any* r*-dimensional subspace. Then there exists an* ΩK*-unimodular map* $M : \Lambda Y \to \Lambda Y$ *such that* $\operatorname{Im} M \cdot \bar{f} = \Lambda Y_0$.

Proof. Let $\mathfrak{T}$ denote the transfer matrix of $\bar{f}$, and let the Smith-McMillan representation of $\mathfrak{T}$ be given as

$$\mathfrak{T} = M_L D M_R,$$

where M_L and M_R are ΩK-unimodular matrices, and $D = \operatorname{diag}(\gamma_1, \dots, \gamma_r, 0, \dots, 0)$ is the Smith-McMillan form of $\mathfrak{T}$. Now, M_L^{-1} is also ΩK-unimodular and, upon identifying maps with their transfer matrices, we obtain

$$\operatorname{Im}(M_L^{-1}\mathfrak{T}) = D M_R \Lambda Y = D \Lambda Y = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Lambda Y.$$

We now define an invertible K-linear map $V: Y \to Y$ such that

$$\operatorname{Im} V \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = Y_0.$$

Then we obtain

$$\operatorname{Im}\left(VM_L^{-1}\mathfrak{T}\right) = V \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Lambda Y = \Lambda Y_0,$$

so the proof is complete upon setting $M := VM_L^{-1}$. ∎

LEMMA 5.3. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map. If $\mathfrak{R} \subset \ker \pi_{\Omega K}\bar{f}$ is a ΛK-linear subspace, then $\mathfrak{R} \subset \ker \bar{f}$.*

Proof. Assume $u \in \mathfrak{R} \subset \ker \pi_{\Omega K}\bar{f}$, where $\mathfrak{R}$ is a ΛK-linear subspace. Then $\alpha u \in \ker \pi_{\Omega K}\bar{f}$ for all $\alpha \in \Lambda K$. Thus $\bar{f}(\alpha u) = \alpha \bar{f}(u) \in \Omega K$ for all $\alpha \in \Lambda K$, whence, since $\Omega K \neq \Lambda K$, necessarily $\bar{f}(u) = 0$ and $u \in \ker \bar{f}$. ∎

Proof of Theorem 5.1. The necessity was already seen at the beginning of the section. To prove the sufficiency, assume that $\ker \pi_{\Omega K}\bar{f}_1 \subset \ker \pi_{\Omega K}\bar{f}_2$. Let $r := \dim_{\Lambda K}\operatorname{Im}\bar{f}_1$, and let Y_0 be any r-dimensional subspace of Y. By Lemma 5.2 there exists an ΩK-unimodular map $M: \Lambda Y \to \Lambda Y$ such that $\operatorname{Im} M\bar{f}_1 = \Lambda Y_0$. Denoting $\bar{f}_0 := M\bar{f}_1$, it follows at once, from the necessity condition above combined with the fact that both M and M^{-1} are ΩK-maps, that $\operatorname{Ker} \pi_{\Omega K}\bar{f}_0 = \ker \pi_{\Omega K}\bar{f}_1$. Thus, $\ker \pi_{\Omega K}\bar{f}_0 \subset \ker \pi_{\Omega K}\bar{f}_2$. Lemma 5.3 then implies that $\ker \bar{f}_0 \subset \ker \bar{f}_2$, so that there exists a ΛK-linear map $\bar{h}_0: \Lambda Y \to \Lambda W$ such that $\bar{h}_0\bar{f}_0 = \bar{f}_2$. We still have to show that $\bar{h}_0$ can be chosen as an ΩK-map. Let $Y_1 \subset Y$ be a direct summand of Y_0 in Y, that is, $Y = Y_0 \oplus Y_1$. Also, let $\bar{P}: \Lambda Y \to \Lambda Y$ denote the projection onto ΛY_0 along ΛY_1, i.e., if $y = y_0 + y_1 \in \Lambda Y$ is the decomposition of y into its components $y_0 \in \Lambda Y_0$ and $y_1 \in \Lambda Y_1$, then $\bar{P}(y) = y_0$. We now define the map $\bar{h} := \bar{h}_0 \cdot \bar{P} \cdot \bar{M}$, and for each $u \in \Lambda U$ we have

$$\bar{h}\bar{f}_1(u) = \bar{h}_0\bar{P}\bar{M}\bar{f}_1(u) = \bar{h}_0\bar{P}\bar{f}_0(u) = \bar{h}_0\bar{f}_0(u) = \bar{f}_2(u), \tag{5.4}$$

whence $\bar{h}\bar{f}_1 = \bar{f}_2$. To conclude the proof, we need to show that $\bar{h}$ is an ΩK-map. Since, by definition, M is an ΩK-map, it suffices to prove that so is also $\bar{h}_0\bar{P}$. To this end, first note that every element $y \in \Omega Y$ decomposes uniquely into $y = y_0 + y_1$ with $y_0 \in \Omega Y_0$ and $y_1 \in \Omega Y_1$. Thus, for every $y \in \Omega Y$, noting that $y_0 \in \bar{f}_1[\ker \pi_{\Omega K}\bar{f}_1]$, we obtain

$$\bar{h}_0\bar{P}(y) = \bar{h}_0(y_0) = \bar{h}_0\bar{P}(y_0) = \bar{h}_0\bar{P}M\bar{f}_1(u) = \bar{h}\bar{f}_1(u) = \bar{f}_2(u)$$

for a suitable $u \in \ker \pi_{\Omega K} \bar{f}_1$. Since by hypothesis $\ker \pi_{\Omega K} \bar{f}_1 \subset \ker \pi_{\Omega K} \bar{f}_2$, it follows that $\bar{h}_p \bar{P}(y) = \bar{f}_2(u) \in \Omega W$, so that $\bar{h}_0 \bar{P}[\Omega Y] \subset \Omega W$, and $\bar{h}_0 \bar{P}$ is an ΩK-map. This completes our proof. ■

Theorem 5.1 admits the following

COROLLARY 5.5. *Let $\bar{f}_1, \bar{f}_2 : \Lambda U \to \Lambda Y$ be ΩK-rational ΛK-linear maps. There exists an ΩK-unimodular map $M: \Lambda Y \to \Lambda Y$ such that $\bar{f}_2 = M\bar{f}_1$ if and only if* $\ker \pi_{\Omega K} \bar{f}_1 = \ker \pi_{\Omega K} \bar{f}_2$.

Proof. The necessity follows immediately from Theorem 5.1. To see the sufficiency, suppose $\ker \pi_{\Omega K} \bar{f}_1 = \ker \pi_{\Omega K} \bar{f}_2$, so that, by Lemma 5.3, also $\ker \bar{f}_1 = \ker \bar{f}_2$ and $\dim \operatorname{Im} \bar{f}_1 = \dim \operatorname{Im} \bar{f}_2 =: r$. Let $Y_0 \subset Y$ be an r-dimensional K-linear subspace, and let $M_1, M_2 : \Lambda Y \to \Lambda Y$ be ΩK-unimodular maps such that $\operatorname{Im} M_1 \bar{f}_1 = \operatorname{Im} M_2 \bar{f}_2 = \Lambda Y_0$ (see Lemma 5.2). Denoting $\bar{f}_{i0} := M_i \bar{f}_i$ $(i = 1,2)$, we obviously still have that $\ker \pi_{\Omega K} \bar{f}_{10} = \ker \pi_{\Omega K} \bar{f}_{20}$. By Theorem 5.1 there are then ΩK-maps $\bar{h}_{10}, \bar{h}_{20} : \Lambda Y \to \Lambda Y$ such that $\bar{f}_{20} = \bar{h}_{10} \bar{f}_{10}$ and $\bar{f}_{10} = \bar{h}_{20} \bar{f}_{20}$. Let $Y_1 \subset Y$ be a direct summand of Y_0 in Y, and let $\bar{P}: \Lambda Y \to \Lambda Y$ be the projection defined in the proof of Theorem 5.1. Now define the ΩK-maps $\bar{h}_1 = \bar{P}(\bar{h}_{10} - I)\bar{P} + I$ and $\bar{h}_2 = \bar{P}(\bar{h}_{20} - I)\bar{P} + I$, where I is the identity map in ΛY. Clearly then also $\bar{f}_{20} = \bar{h}_1 \bar{f}_{10}$ and $\bar{f}_{10} = \bar{h}_2 \bar{f}_{20}$ and also $\bar{h}_2 \bar{h}_1 = \bar{h}_1 \bar{h}_2 = I$, as can be verified by direct computation. It follows that $\bar{h}_1$ is ΩK-unimodular, and the ΩK-unimodular map $M := M_2^{-1} \bar{h}_1 M_1$ satisfies the condition of the corollary. ■

We call a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ *ΩK-left invertible* if it has an ΩK-map as a left inverse, that is, if there exists an ΩK-map $\bar{h}: \Lambda Y \to \Lambda U$ such that $\bar{h}\bar{f} = I$. The following further corollary to Theorem 5.1 characterizes the ΩK-left invertible maps.

COROLLARY 5.6. *An ΩK-rational ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is ΩK-left invertible if and only if* $\ker \pi_{\Omega K} \bar{f} \subset \Omega U$.

Proof. First note that $\ker \pi_{\Omega K} I = \Omega U$, where $I: \Lambda U \to \Lambda U$ is the identity map. If $\bar{f}$ has an ΩK-left inverse $\bar{h}: \Lambda Y \to \Lambda U$ (i.e. $\bar{h}\bar{f} = I$), then $\ker \pi_{\Omega K} \bar{f} \subset \ker \pi_{\Omega K} \bar{h}\bar{f} = \Omega U$. Conversely, if $\ker \pi_{\Omega K} \bar{f} \subset \Omega U$ $(= \ker \pi_{\Omega K} I)$, then the existence of $\bar{h}$ is ensured by Theorem 5.1. ■

Before concluding this section, we wish to express in an explicit form the main quantities that appeared in our discussion. Let $\bar{f}: \Lambda U \to \Lambda Y$ be an ΩK-rational ΛK-linear map. We start with an explicit representation of the ΩK-module $\ker \pi_{\Omega K} \bar{f}$. We shall identify the map $\bar{f}$ with its transfer matrix, and shall denote $r := \dim_{\Lambda K} \operatorname{Im} \bar{f}$. Let $M_L : \Lambda Y \to \Lambda Y$ and $M_R : \Lambda U \to \Lambda U$ be ΩK-unimodular maps such that $\bar{f} = M_L D M_R$, where the matrix $D : \Lambda U \to \Lambda Y$ is of the form

$$D = \begin{pmatrix} D_0 & 0 \\ 0 & 0 \end{pmatrix},$$

with $D_0 : \Lambda K' \to \Lambda K'$ (square) nonsingular. One possible choice of D is, of course, the Smith-McMillan canonical form of $\bar{f}$. Also, we let $U_0 \oplus U_1 = U$ be a direct sum decomposition, where $\Lambda U_0 = \ker D$ and ΛU_1 is the domain of D_0.

Now, $\ker \pi_{\Omega K} \bar{f} = \ker \pi_{\Omega K} M_L D M_R = M_R^{-1}[\ker \pi_{\Omega K} M_L D]$, and, applying Corollary 5.5, we obtain that $\ker \pi_{\Omega K} \bar{f} = M_R^{-1}[\ker \pi_{\Omega K} D]$. Further, it is readily seen that $\ker \pi_{\Omega K} D = D_0^{-1}[\Omega U_1] \oplus \Lambda U_0$ and consequently we have

$$\ker \pi_{\Omega K} \bar{f} = M_R^{-1}\left[D_0^{-1}[\Omega U_1] \oplus \Lambda U_0\right] \tag{5.7}$$

and

$$\ker \bar{f} = M_R^{-1}[\Lambda U_0]. \tag{5.8}$$

Defining now the map

$$\bar{f}_* = M_R^{-1}\begin{pmatrix} D_0^{-1} \\ 0 \end{pmatrix} : \Lambda U_1 \to \Lambda U,$$

we have that

$$\ker \pi_{\Omega K} \bar{f} = \bar{f}_*[\Omega U_1] + \ker \bar{f}, \tag{5.9}$$

so that $\bar{f}_*$ generates the "bounded part" of $\ker \pi_{\Omega K} \bar{f}$.

Next, let $\bar{f}' : \Lambda U \to \Lambda Y'$ be a ΛK-linear map. We express in explicit matrix form the condition of Theorem 5.1. The inclusion $\ker \pi_{\Omega K} \bar{f} \subset \ker \pi_{\Omega K} \bar{f}'$ is evidently equivalent to $\bar{f}'[\ker \pi_{\Omega K} \bar{f}] \subset \Omega Y'$. Substituting now (5.9), and noting that $\ker \bar{f}$ is a ΛK-linear subspace, the latter condition can be split into the two conditions: (i) $\bar{f}'\bar{f}_*[\Omega U_1] \subset \Omega Y'$, (ii) $\bar{f}'[\ker \bar{f}] = 0$. These conditions are then, respectively, equivalent to simply (ia) $\bar{f}\bar{f}_*$ *is an* ΩK*-map, and* (iia) $\ker \bar{f} \subset \ker \bar{f}'$.

Returning now to Theorem 5.1, we can summarize as follows: *There exists an* ΩK*-map* $\bar{h} : \Lambda Y \to \Lambda Y$ *such that* $\bar{f}' = \bar{h}\bar{f}$ *if and only if* $\bar{f}'\bar{f}_*$ *is an* ΩK*-map* *and* $\ker \bar f \subset \ker \bar f'$. Moreover, through a direct computation one can show that, if $\bar h$ exists, then it is necessarily of the form

(5.10) $$\bar h = \left(\bar f' \bar f_*, y_1, \dots, y_{p-r}\right) M_L^{-1},$$

where $p := \dim_K Y$, and $y_1, \dots, y_{p-r}$ are (arbitrary) elements in $\Omega Y'$. Thus, the map $\bar f_*$, which generates the "bounded part" of $\ker \pi_{\Omega K} \bar f_1$, plays a central role in factorization theory, serving as a certain generalized type of "inverse" of $\bar f$ for the purpose of explicit ΩK-factorization.

6. PRECOMPENSATION AND STABLE OUTPUT FEEDBACK

We turn now to a brief discussion of some applications of the above factorization theory to the study of feedback systems, in which the feedback compensator is stable (and causal). Let $\bar f: \Lambda U \to \Lambda Y$ be a linear i/o map (with U and Y finite dimensional), and let $\bar l: \Lambda U \to \Lambda U$ be a bicausal ΛK-linear map (i.e., $\Omega^- K$-unimodular) which we regard as a precompensator for $\bar f$. We can express $\bar l^{-1}$ as

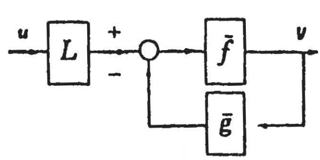
(6.1) $$\bar l^{-1} = L^{-1}(I + \bar h),$$

where $L: U \to U$ is static [6] and where $\bar h$ is strictly causal. If, additionally, we can express $\bar h$ as $\bar h = \bar g \bar f$ for some causal map $\bar g: \Lambda Y \to \Lambda U$, then we can give $\bar l$ an output feedback interpretation through the formula

$$\bar f \bar l = \bar f (I + \bar g \bar f)^{-1} L,$$

which can be represented as the following block diagram:

(6.2)



The map $\bar g$ is then clearly a causal dynamic output feedback compensator, and L is a coordinate transformation map in the input value space. The problem of representing a precompensator $\bar l$ as a configuration of the form (6.2) is considered in [3]. In the present section we consider this problem under the additional requirement that the feedback compensator $\bar{g}$ be *stable*. From the applications point of view, it is, of course, preferable to deal with stable compensators, whenever this is possible. Clearly, the feedback compensator can be chosen as a stable (and causal) system exactly when the map $\bar{h}$ of (6.1) can be factored over $\bar{f}$ through an $\Omega_{\mathfrak{T}}K$-map $\bar{g}$. Using Theorem 5.1, we arrive at the following.

THEOREM 6.3. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a rational linear i/o map, let $\bar{l}: \Lambda U' \to \Lambda U$ be a rational bicausal precompensator for $\bar{f}$, and express $\bar{l}$ as in (6.1). There exists a causal and stable output feedback representation for $\bar{l}$ if and only if* $\ker \pi_{\Omega_{\mathfrak{T}}K} \bar{f} \subset \ker \pi_{\Omega_{\mathfrak{T}}K} \bar{h}$.

REMARK 6.4. A system is said to be *internally stable* if all its modes, including the unreachable and the unobservable ones, are stable. The notion of internal stability is particularly important when considering composite systems, since the composition may generate hidden modes, and if these are unstable, the stability of the final system will of course be destroyed. We are presently interested in the composite system (6.2). It can be shown that (6.2) is internally stable if and only if all four of the maps, $\bar{l}$, $\bar{f}\bar{l}$, $\bar{l}\bar{g}$, and $\bar{f}\bar{l}\bar{g}$ are i/o stable [3]. In particular, in the case of *stable* feedback, $\bar{g}$ is stable, and we obtain that (6.2) is internally stable if and only if both of the maps $\bar{l}$ and $\bar{f}\bar{l}$ are i/o stable.

We say that a linear i/o map $\bar{f}: \Lambda U \to \Lambda Y$ is *$\Omega_{\mathfrak{T}}K$-minimum phase* (or, simply, *minimum phase*) if it is an $\Omega_{\mathfrak{T}}K$-map (i.e., stable) and is $\Omega_{\mathfrak{T}}K$-left invertible. Thus, using Proposition 2.4 and Corollary 5.6, we obtain that $\bar{f}$ is $\Omega_{\mathfrak{T}}K$-minimum phase precisely whenever

(6.5) $$\ker \pi_{\Omega_{\mathfrak{T}}K} \bar{f} = \Omega_{\mathfrak{T}} U.$$

We recall further [5] that a linear i/o map $\bar{f}$ is called *nonlatent* if

(6.6) $$\ker \pi_{\Omega^- K} \bar{f} = z\Omega^- U.$$

Clearly, (6.6) is equivalent to $\ker \pi_{\Omega^- K}(z\bar{f}) = \Omega^- U$. Thus, by Proposition 2.4 and Corollary 5.6, $\bar{f}$ is nonlatent if and only if $z\bar{f}$ is both causal and $\Omega^- K$-left invertible. Obviously, in the last statement, $z\bar{f}$ can be replaced by $(z+\alpha)\bar{f}$, where α is any element in the field K.

In particular, assume that $z+\alpha$ is in the denominator set $\mathfrak{T}$. Then, clearly, $\bar{f}$ is minimum phase if and only if $(z+\alpha)\bar{f}$ is. Combining now (6.5) and (6.6), we obtain that $\bar{f}$ *is both nonlatent and minimum phase if and only if*

$$\ker \pi_{\Omega_{\mathfrak{D}}^{-}K}\bar{f} = (z+\alpha)\Omega_{\mathfrak{D}}^{-}U, \tag{6.7}$$

where $z + \alpha \in \mathfrak{D}$.

We now have the following theorem which is an analog to Corollary 5.4 in [5].

THEOREM 6.8. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be an i/o-stable linear i/o map. Assume that the denominator set $\mathfrak{D}$ contains two different first degree polynomials $z+\alpha$ and $z+\beta$. Then $\bar{f}$ is nonlatent and minimum phase if and only if every $\Omega_{\mathfrak{D}}^{-}K$-unimodular precompensator $\bar{l}: \Lambda U \to \Lambda U$ has a causal and stable feedback representation $(L, \bar{g})$, i.e., there exists a pair $(L, \bar{g})$, where L is static and $\bar{g}$ is causal and i/o stable, such that $\bar{l} = [I + \bar{g}\bar{f}\,]^{-1}L$.*

Proof. If $\bar{f}$ is nonlatent and minimum phase, then

$$\ker \pi_{\Omega_{\mathfrak{D}}^{-}K}\bar{f} = z\Omega_{\mathfrak{D}}^{-}U \subset \ker \pi_{\Omega_{\mathfrak{D}}^{-}K}\bar{h}$$

for every strictly causal and stable $\bar{h}$. Hence sufficiency holds, and $\bar{l}\,[=(I+\bar{h})^{-1}L]$ has a causal and stable feedback representation. Conversely, assume that every $\Omega_{\mathfrak{D}}^{-}K$-unimodular $\bar{l}$ has a causal and stable feedback representation $(L, \bar{g})$. In particular, consider the $\Omega_{\mathfrak{D}}^{-}K$-unimodular precompensator

$$\bar{l} := \frac{z+\alpha}{z+\beta}I: \Lambda U \to \Lambda U,$$

where $z+\alpha, z+\beta \in \mathfrak{D}$ and $\alpha \neq \beta$. Then, denoting $\gamma := \beta - \alpha$, we obtain that $\bar{l} = [I + \bar{h}]^{-1}$, where $\bar{h} = [\gamma/(z+\alpha)]I$. Now, by assumption, there exists a causal and i/o-stable map $\bar{g}$ such that $\bar{h} = \bar{g}\bar{f}$, whence, by Theorem 5.1, $\ker \pi_{\Omega_{\mathfrak{D}}^{-}K}\bar{f} \subset \ker \pi_{\Omega_{\mathfrak{D}}^{-}K}\bar{h} = (z+\alpha)\Omega_{\mathfrak{D}}^{-}U$. Furthermore, since $\bar{f}$ is strictly causal and i/o stable, also $(z+\alpha)\Omega_{\mathfrak{D}}^{-}U \subset \ker \pi_{\Omega_{\mathfrak{D}}^{-}K}\bar{f}$, and we obtain that $\ker \pi_{\Omega_{\mathfrak{D}}^{-}K}\bar{f} = (z+\alpha)\Omega_{\mathfrak{D}}^{-}U$. Thus, by (6.7), $\bar{f}$ is nonlatent and minimum phase. ■

The interest in Theorem 6.8 derives from the fact that stable injective linear i/s maps [6] are always nonlatent and minimum phase. This fact is seen as follows. It was shown in [4] that if $\bar{f}: \Lambda U \to \Lambda Y$ is an injective linear i/s map, it is *strictly observable*, i.e. $\ker \pi_{\Omega^{+}K}\bar{f} \subset \Omega^{+}U$. Let D be an $\Omega^{+}K$-adapted basis matrix for $\ker \pi_{\Omega^{+}K}\bar{f}$, that is, $D\Omega^{+}U = \ker \pi_{\Omega^{+}K}\bar{f}$. It is easily verified that we then also have that $D\Omega_{\mathfrak{D}}U = \ker \pi_{\Omega_{\mathfrak{D}}K}\bar{f}$. Now, the strict observability of $\bar{f}$ implies that D is a polynomial matrix, and thus $D\Omega_{\omega}U \subset \Omega_{\omega}U$ (since $\Omega^{+}K \subset \Omega_{\omega}K$). We conclude that $\ker \pi_{\Omega_{\omega}K}\bar{f} \subset \Omega_{\omega}U$, and if the i/s map $\bar{f}$ is also stable, the minimum phase property [see (6.5)] follows. That injective linear i/s maps are nonlatent was proved in [5, Theorem 5.5]. We summarize the above in the following

PROPOSITION 6.9. *If $\bar{f}: \Lambda U \to \Lambda Y$ is a stable injective linear i/s map, then it is nonlatent and minimum phase.*

We can now combine Theorem 6.8 with Proposition 6.9 to obtain the following interesting result.

THEOREM 6.10. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a stable, injective linear i/o map, and let $\bar{l}: \Lambda U \to \Lambda U$ be an $\Omega_{\bar{\omega}}K$-unimodular precompensator for $\bar{f}$. Then $\bar{l}$ has a stable causal (dynamic) state feedback representation in every stable realization of $\bar{f}$.*

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