Math. Systems Theory 17, 135–157 (1984)

Mathematical Systems Theory

Decoupling of Linear Systems by Dynamic Output Feedback[1]

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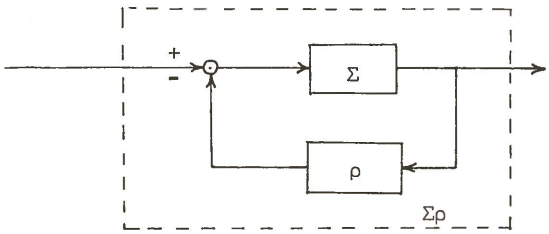
Abstract. Necessary and sufficient conditions for decoupling of linear systems by dynamic output feedback are derived, under the requirement that the decoupled system be internally stable. The conditions are stated in terms of quantities which are directly related to the transfer matrix of the given system. The main issue is resolved through the introduction of a new concept —the strict adjoint. The strict adjoint is a "minimal" polynomial matrix that diagonalizes a given matrix.

1. Introduction

The problem of decoupling through the employment of dynamic output feedback is formulated as follows. Let Σ be a linear time invariant system, and consider the following diagram



(1.1)

where ρ is, again, a linear time invariant system (the *output feedback*), and Σ_ρ denotes the depicted combination. We say that the system Σ_ρ is *decoupled* if its transfer matrix is diagonal.

Our objective in the present paper is to formulate necessary and sufficient conditions (on Σ) for the existence of a feedback ρ such that Σ_ρ is decoupled. A major issue in the present situation is the necessity to ensure the "internal

[1]This research was supported in part by US Army Research Grant DAAG29-80-C0050 and US Air Force Grant AFOSR76-3034D through the Center for Mathematical System Theory, University of Florida, Gainesville, Florida 32611, U.S.A.

stability" of Σ_p, that is, the complete stability of all its modes, including the unobservable and uncontrollable ones. For the sake of simplicity of presentation, we confine most of our discussion to the case where the transfer matrix of Σ is nonsingular and square.

It turns out that two new concepts—"strict adjoint" and "d-coprimeness"—play a central role in our solution of the decoupling problem. Intuitively speaking, the (right) strict adjoint of a polynomial matrix P is a "minimal" polynomial matrix P_* such that PP_* is diagonal. (This notion is precisely defined in Section 2.) Since decoupled systems have diagonal transfer matrices, it is not surprising that the concept of strict adjoint is very useful in analyzing the decoupling problem. The other notion—that of d-coprimeness—is a stronger version of the classical matrix coprimeness conditions. As is well known, two polynomial matrices A and B are right coprime if and only if there exist polynomial matrices Y_1 and Y_2 such that $Y_1A+Y_2B=I$. For d-coprimeness, the matrix Y_1 is required to be diagonal. In principle, it is computationally easier to verify d-coprimeness than usual coprimeness, since fewer free parameters are involved.

In the main section of the paper (Section 3), we state necessary and sufficient conditions for the decoupling of a given square and nonsingular transfer matrix f, under the requirement that the resulting decoupled system be internally stable. We also single out there several cases in which the decoupling conditions are particularly simple. As in many other problems related to output feedback control, here too the unstable zeros of the given system f play an important role. In particular, in case f has no unstable zeros, the decoupling conditions become very simple, and f can be decoupled if and only if the polynomial part of f^{-1} is a diagonal matrix (see Corollary 3.8).

The problem of decoupling linear time invariant systems received considerable attention in the system theoretic literature during the past two decades. Much of this attention was directed toward decoupling through the employment of state feedback (Morgan [1964], Rekasius [1965], Falb and Wolovich [1967], Gilbert [1969], Hautus and Heymann [1980]). More generalized schemes, allowing the decoupled system to consist of multi-input multi-output "blocks", and also allowing state space extension, were investigated by Wonham and Morse [1970], Basile and Marro [1970], Morse and Wonham [1970], and Wonham [1974]. Conditions for decoupling through the employment of a combination of both state feedback and dynamic precompensation were formulated by Silverman [1970], and Silverman and Payne [1971], and decoupling through a combination of precompensation and output feedback was considered by Pernebo [1981].

An additional important class of decoupling problems is related to the use of static (constant gain) output feedback. Necessary and sufficient conditions for decoupling through static output feedback were given in Falb and Wolovich [1967], Howze [1973], Wang and Davison [1975], and Wolovich [1975]. Finally, the question of decoupling, using combinations of dynamic compensation and static output feedback, was considered by Howze and Pearson [1970].

We divide our following discussion into two sections, the first of which is devoted to a survey and development of a suitable mathematical framework, and the last of which includes the decoupling conditions. An appendix provides explicit constructions for the actual verification of the decoupling conditions.

2. Terminology and Preliminaries

Let K be a field, and let Σ be a K-linear time invariant system. We denote by U the K-linear space of input values to Σ, and let Y be the K-linear space where the output values of Σ occur. Throughout our discussion we assume that the K-linear spaces U and Y are finite dimensional, and let $U=K^m$ and $Y=K^p$. We also assume that Σ admits a finite dimensional realization. The transfer matrix of Σ can then be regarded as a linear map between certain linear spaces of sequences, as follows.

Let S be a K-linear space, and let ΛS denote the set of all Laurent series in the indeterminate z^{-1}, of the form

$$s = \sum_{t=t_0}^{\infty} s_t z^{-t}, \tag{2.1}$$

where, for all t, $s_t \in S$. It can then be readily verified that, under coefficientwise addition and convolution as scalar multiplication, the set ΛK is endowed with a field structure, and ΛS forms a ΛK-linear space. Moreover, when the K-linear space S is finite dimensional, so is also ΛS as a ΛK-linear space, and $\dim_{\Lambda K}\Lambda S = \dim_K S$. We note that the field ΛK contains, as subsets, the set Ω^+K of all (polynomial) elements of the form $\sum_{t\leqq 0} k_t z^{-t}$, and the set Ω^-K of all (power series) elements of the form $\sum_{t\geqq 0} k_t z^{-t}$. Both of Ω^+K and Ω^-K are endowed with a principal ideal domain structure under the operations of addition and multiplication defined in ΛK.

Returning now to the system Σ, we see that the transfer matrix of Σ has its entries in ΛK, and thus induces a ΛK-linear map $f\colon \Lambda U \to \Lambda Y$. In case Σ is a discrete time system, every element $u = \sum_{t=t_0}^{\infty} u_t z^{-t} \in \Lambda U$ can be interpreted as an input time sequence to Σ, with the index t being indentified as the time marker. In this case, $fu \in \Lambda Y$ corresponds to the output sequence of Σ, generated by the input sequence u (see also Wyman [1972]). Below, we identify the transfer matrix of Σ with its corresponding ΛK-linear map. Conversely, every ΛK-linear map $f\colon \Lambda U \to \Lambda Y$ can, of course, be represented as a matrix relative to specified bases $u_1,\dots,u_m$ of ΛU and $y_1,\dots,y_p$ of ΛY. In particular, if $u_1,\dots,u_m$ are in U and $y_1,\dots,y_p$ are in Y (where U and Y are regarded as subsets of ΛU and ΛY, respectively), then the matrix representation of f is called a *transfer matrix*. It can be readily seen that this terminology is consistent with our previous discussion.

Several particular types of ΛK-linear maps will be important to us below, and we now proceed to examine some of their properties. First, given a ΛK-linear map $f\colon \Lambda U \to \Lambda Y$, we say that f is *polynomial* if its transfer matrix is a polynomial matrix. Also, f is called *rational* if there exists a nonzero polynomial ψ such that ψf is polynomial. Next, we review the concept of causality. To this end, we assign to each element $s = \sum_{t=t_0}^{\infty} s_t z^{-t} \in \Lambda S$ an integer, called the *order* of s, as follows:

$\operatorname{ord} s := \min_t \{s_t \neq 0\}$ if $s \neq 0$, and $\operatorname{ord} s := \infty$ if $s = 0$. The *leading coefficient* $\hat{s}$ of s is defined as $\hat{s} := s_{\operatorname{ord} s}$ if $s \neq 0$ and $\hat{s} := 0$ if $s = 0$. Then, a ΛK-linear map $f: \Lambda U \to \Lambda Y$ is called *causal* (respectively *strictly causal*) whenever $\operatorname{ord} fu \geq \operatorname{ord} u$ (respectively $\operatorname{ord} fu > \operatorname{ord} u$), for all $u \in \Lambda U$. A ΛK-linear map $l: \Lambda U \to \Lambda U$ is called *bicausal* if l has an inverse l^{-1}, and if both of l and l^{-1} are causal. Finally, a strictly causal and rational ΛK-linear map $f: \Lambda U \to \Lambda Y$ is called a *linear input/output map* (Hautus and Heymann [1978]).

Much of our attention in the present paper is devoted to stability considerations, and we next set up our terminology in this context. Let $\sigma \subset \Omega^+ K$ be a multiplicative polynomial set (i.e. for every pair of elements $k_1, k_2 \in \sigma$, also $k_1 k_2 \in \sigma$). We say that σ is a *stability set* if both $0 \notin \sigma$ and there exists an element $\alpha \in K$ such that $(z + \alpha) \in \sigma$ (Morse [1975]). We shall use the letter σ to denote a (fixed) stability set throughout our discussion. Now, a ΛK-linear map $f: \Lambda U \to \Lambda Y$ is called *input/output stable* (in the sense of σ) if there exists an element $\psi \in \sigma$ such that ψf is a polynomial map. Evidently, in case K is the field of real numbers, input/output stability includes the classical notion of stability in linear control theory, where all the system poles are required to lie within a specified region of the complex plane (intersecting the real line). The notion of input/output stability can be naturally accommodated in the framework of the theory of matrices having their entries in a principal ideal domain. Indeed, let $\Omega_\sigma^+ K$ denote the set of all elements $\alpha \in \Lambda K$ which can be expressed as a polynomial fraction $\alpha = \beta/\gamma$, with denominator γ belonging to σ (i.e. the set of all input/output stable elements in ΛK). Then, $\Omega_\sigma^+ K$ forms a principal ideal domain under the operations defined in ΛK (see, e.g. Hammer [1983a]), and a ΛK-linear map $f: \Lambda U \to \Lambda Y$ is input/output stable if and only if its transfer matrix has all its entries in $\Omega_\sigma^+ K$. We shall use the following terminology, which is in accordance with classical terms in the theory of matrices (e.g. Macduffee [1934]). First, a ΛK-linear map $M: \Lambda U \to \Lambda U$ is called $\Omega_\sigma^+ K$*-unimodular* if it possesses an inverse M^{-1}, and if both of M and M^{-1} are input/output stable. Further, let $f: \Lambda U \to \Lambda Y$ and $g: \Lambda U \to \Lambda Y'$ be input/output stable ΛK-linear maps. An input/output stable ΛK-linear map $h: \Lambda U \to \Lambda U$ is a *common right σ-divisor* of f and g if there exist input/output stable maps $f': \Lambda U \to \Lambda Y$ and $g': \Lambda U \to \Lambda Y'$ such that $f = f'h$ and $g = g'h$. The maps f and g are *right σ-coprime* if all their common right σ-divisors are $\Omega_\sigma^+ K$-unimodular. Equivalently, f and g are right σ-coprime if and only if there exist input/output stable maps $\alpha: \Lambda Y \to \Lambda U$ and $\beta: \Lambda Y' \to \Lambda U$ such that $\alpha f + \beta g = I$, the identity matrix (see Macduffee [1934, Chapter 3]). Dually, one defines, in an evident way, *common left σ-divisors* and *left σ-coprime* maps.

The use of σ-coprime maps allows certain representations of systems, which are useful in stability considerations. Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. A representation $f = PQ^{-1}$, where $Q: \Lambda U \to \Lambda U$ and $P: \Lambda U \to \Lambda Y$ are input/output stable ΛK-linear maps, is called a (right) *stability representation*. A stability representation $f = PQ^{-1}$ is called *canonical* if P and Q are right σ-coprime. Then, one can show (see Hammer [1983a]) that *every rational ΛK-linear map $f: \Lambda U \to \Lambda Y$ possesses a canonical (right) stability representation $f = PQ^{-1}$. In this representation, f is input/output stable if and only if Q is $\Omega_\sigma^+ K$-unimodular.* Canonical left stability representations are, of course, dual. In the following two theorems (reproduced from Hammer [1983a]), we establish the existence of two particular types of stability representations, one of which is related to the unstable zeros of the system, and the other—to its unstable poles. We shall describe the explicit construction of these representations in a moment.

Theorem 2.2. *Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then there exists a (right) canonical stability representation $f = PQ^{-1}$ satisfying the following:* (i) *P is polynomial, and* (ii) *if $f = P_1 Q_1^{-1}$ is any stability representation with P_1 a polynomial matrix, then P is a (polynomial) left divisor of P_1.*

A stability representation satisfying conditions (i) and (ii) of Theorem 2.2 is called a *canonical zero representation* (of f). We note that the matrices P and Q in this representation are determined up to a polynomial unimodular right multiplier.

Theorem 2.3. *Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then, there exists a (right) canonical stability representation $f = PQ^{-1}$ satisfying the following:* (i) *Q is polynomial, and* (ii) *if $f = P_1 Q_1^{-1}$ is any stability representation with Q_1 a polynomial matrix, then Q is a (polynomial) left divisor of Q_1.*

Again, a stability representation satisfying conditions (i) and (ii) of Theorem 2.3 is called a *canonical pole representation* (of f).

A canonical zero representation is explicitly constructed as follows. Let $f\colon \Lambda U \to \Lambda Y$ be a nonzero ΛK-linear map, and let $f = ND^{-1}$ be a right coprime polynomial matrix fraction representation. Also, let $M_1\colon \Lambda Y \to \Lambda Y$ and $M_2\colon \Lambda U \to \Lambda U$ be polynomial unimodular matrices such that the matrix $\delta := M_1 N M_2$ is in Smith canonical form, say $\delta = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_r, 0, \ldots, 0)$, where $\delta_i \neq 0$ for all $i = 1, \ldots, r$. For all $i = 1, \ldots, r$, we factor now $\delta_i = \delta_i' \delta_i''$ into a multiple of polynomials, where δ_i' is coprime with every element in the stability set σ, and $1/\delta_i'' \in \Omega_\sigma^+ K$. Now, with the $p \times m$ diagonal matrix $\delta' := \operatorname{diag}(\delta_1', \delta_2', \ldots, \delta_r', 0, \ldots, 0)\colon \Lambda U \to \Lambda Y$, and the $m \times m$ diagonal matrix $\delta'' := \operatorname{diag}(\delta_1'', \delta_2'', \ldots, \delta_r'', 1, \ldots, 1)\colon \Lambda U \to \Lambda U$, we let $P := M_1 \delta'$ and $P_1 := \delta'' M_2$, so that $N = PP_1$, and P_1 is nonsingular with stable inverse. Then, defining $Q := DP_1^{-1}$, it can be shown that $f = PQ^{-1}$ is a canonical zero representation of f. The factorization $N = PP_1$ is actually a somewhat weaker form of the classical left standard factorization of N (see the different factorizations considered by Gokhberg and Krein [1960], and Youla [1961]). The matrix P, which characterizes the unstable zeros of f, is called a *zero matrix*. The matrix P was also employed in Pernebo [1981], where it was called a "left structure matrix".

Next, we describe the explicit construction of a canonical pole representation. Again, let $f = ND^{-1}$ be a right coprime polynomial matrix fraction representation. We factor $D = QQ_1$, where Q and Q_1 are polynomial matrices; the invariant factors of Q are coprime with every element in σ; and Q_1^{-1} is input/output stable. Then, letting $R := NQ_1^{-1}$, it can be shown that $f = RQ^{-1}$ is a canonical (right) pole representation of f.

Theorems 2.2 and 2.3 lead to the following definition. Let $Q\colon \Lambda U \to \Lambda U$ be a nonsingular polynomial matrix. We say that Q is *completely unstable* if the invariant factors of Q are coprime with every polynomial ψ in σ. It can be readily seen that the matrix Q in Theorem 2.3 is completely unstable, and so is also P in Theorem 2.2. Evidently, every polynomial divisor of a completely unstable matrix is completely unstable as well, a fact that implies the following instrumental

Lemma 2.4. *Let $P, Q: \Lambda U \to \Lambda U$ be polynomial matrices, and assume that P is nonsingular and completely unstable. If $f := P^{-1}Q$ is input/output stable, then f is polynomial.*

We now turn to a discussion of feedback. In diagram (1.1), let the system Σ be represented by the linear input/output map $f: \Lambda U \to \Lambda Y$, and let the causal rational ΛK-linear map $r: \Lambda Y \to \Lambda U$ represent the feedback ρ. The resulting system Σ_ρ is then, again, represented by a linear input/output map $f_r: \Lambda U \to \Lambda Y$ given by the equation

$$f_r = fl_r, \tag{2.5}$$

where

$$l_r := [I + rf]^{-1}: \Lambda U \to \Lambda U \tag{2.6}$$

is a bicausal ΛK-linear map. (The fact that l_r is bicausal is implied by the strict causality of f and the causality of r.)

In Hammer [1983b], a full invariant under the operation of linear dynamic output feedback was derived. In our present discussion we shall employ this invariant in the particular case of systems having square nonsingular transfer matrices, and we specialize now the discussion there to this case. Let $D: \Lambda U \to \Lambda U$ be any matrix, and let $d_i = \sum_{t=t_0}^{\infty} d_t^i z^{-t}$ be the i-th column of D. We denote by $D|(0)$ the strictly polynomial part of D, namely, the matrix consisting of the columns $\sum_{t<0} d_t^i z^{-t}$, $i = 1, \ldots, m$. Then, by Hammer [1983b] Corollary 3.11, we obtain the following

Theorem 2.7. *Let $f, f': \Lambda U \to \Lambda U$ be nonsingular linear input/output maps. Then, there exists an input/output map $r: \Lambda U \to \Lambda U$ such that $f' = f_r$ if and only if $f'^{-1}|(0) = f^{-1}|(0)$. Moreover, $f^{-1}|(0)$ is nonsingular and there exists a causal ΛK-linear map $r_0: \Lambda U \to \Lambda U$ such that $f_{r_0} = [f^{-1}|(0)]^{-1}$.*

Of major importance to the theory of linear dynamic output feedback is the notion of internal stability. Informally, a system is said to be internally stable if all its modes, including the unreachable and unobservable ones, are stable. Various explicit conditions for internal stability of feedback configurations have been stated in the literature (see Desoer and Chan [1975]). For our present discussion, the following set of conditions is convenient (Hammer [1983b]).

Theorem 2.8. *Let $f: \Lambda U \to \Lambda Y$ be a linear input/output map, and let $r: \Lambda Y \to \Lambda U$ be a causal ΛK-linear map. Denote $l_r := [I + rf]^{-1}$. Then, the feedback configuration f_r is internally stable if and only if all of f_r, l_r, $f_r r$ and $l_r r$ are input/output stable.*

We shall also need the following (see also Hammer [1983a]).

Corollary 2.9. *Let $f: \Lambda U \to \Lambda Y$ be a linear input/output map, and let $f = PQ^{-1}$ be a canonical stability representation. Let $r: \Lambda Y \to \Lambda U$ be an input/output map,* *and denote* $h := Q^{-1}l_r$. *Then,* f_r *is internally stable if and only if both of* h *and* hr *are input/output stable.*

Proof. Assume first that h and hr are input/output stable. Then, since clearly $f_r = Ph$, $f_r r = Phr$, $l_r = Qh$, and $l_r r = Qhr$, it follows that all of f_r, $f_r r$, l_r, and $l_r r$ are input/output stable, so that f_r is internally stable by Theorem 2.8. Conversely, assume that f_r is internally stable. Then, by Theorem 2.8, all of f_r, $f_r r$, l_r, and $l_r r$ are input/output stable. Now, since P and Q are right σ-coprime, there exist input/output stable maps $A, B: \Lambda U \to \Lambda U$ such that $AP + BQ = I$. But then, since $h = Af_r + Bl_r$ and $hr = Af_r r + Bl_r r$, it follows that h and hr are input/output stable. □

Finally, the conditions for internal stability can also be stated in the following compact form

Proposition 2.10. *Let* $f: \Lambda U \to \Lambda Y$ *be a linear input/output map, and let* $r: \Lambda Y \to \Lambda U$ *be a causal* ΛK*-linear map. Let* $f = PQ^{-1}$ *and* $r = D^{-1}N$ *be canonical (right and left) stability representations, and denote* $D_1 := l_r^{-1}Q$. *Then,* f_r *is internally stable if and only if* DD_1 *is* $\Omega_\sigma^+ K$*-unimodular.*

Proof. We denote $A := DD_1$. Now, since $A = D[I + rf]Q = DQ + NP$, A is input/output stable by definition. Next, by Corollary 2.9, f_r is internally stable if and only if both D_1^{-1} and $D_1^{-1}r$ are input/output stable, that is, if and only if both of $A^{-1}D$ and $A^{-1}N$ are input/output stable. But, since D, N are input/output stable and left σ-coprime, the latter conditions hold if and only if A^{-1} is input/output stable (See the argument used in proof of Corollary 2.9). Thus, f_r is internally stable if and only if A is $\Omega_\sigma^+ K$-unimodular. □

We note that Proposition 2.10 has a simple intuitive interpretation. Namely, f_r is internally stable if and only if all the "unstable zeros" of l_r are either "poles" of f or "poles" of r. This last statement is, of course, very vague, and we bring it here only for intuitive insight. In applications, the exact statement of the proposition has to be used.

The main topic of our discussion in the present paper, namely, the decoupling problem, involves the study of diagonalization of polynomial matrices. Crucial in this study is the following concept. Let $P, Q: \Lambda U \to \Lambda U$ be nonsingular polynomial matrices. We say that P is a *left strict adjoint* of Q if the following conditions are satisfied: (i) PQ is diagonal, and (ii) if $R: \Lambda U \to \Lambda U$ is any polynomial nonsingular matrix such that RQ is diagonal, then P is a right divisor of R. Intuitively, a strict adjoint is a "minimal polynomial diagonalizer". We define a right strict adjoint in a dual way.

Lemma 2.11. *Let* $P: \Lambda U \to \Lambda U$ *be a nonsingular polynomial matrix. Then,* P *has both a left and a right strict adjoint.*

Proof. We state the proof for the case of right strict adjoints; the other case is dual. For all $i = 1, \ldots, m$, let $(P^{-1})_i$ be the i-th column of P^{-1}, and let (γ_i) be the (polynomial) ideal of all elements $\delta \in \Omega^+ K$ such that $\delta(P^{-1})_i$ is a polynomial vector. Then, the polynomial matrix $Q := P^{-1}\mathrm{diag}(\gamma_1, \ldots, \gamma_m)$ evidently satisfies

$PQ = \mathrm{diag}(\gamma_1,\dots,\gamma_m)$. Moreover, let $R: \Lambda U \to \Lambda U$ be any nonsingular polynomial matrix such that PR is diagonal and polynomial, and let $PR = \mathrm{diag}(\delta_1,\dots,\delta_m)$. Then $P^{-1}\mathrm{diag}(\delta_1,\dots,\delta_m)(=R)$ is polynomial, so that $\delta_i \in (\gamma_i)$ for all $i = 1,\dots,m$. Hence, $\alpha_i := \delta_i/\gamma_i$ is a polynomial for all $i = 1,\dots,m$, so that $R = Q\,\mathrm{diag}(\alpha_1,\dots,\alpha_m)$, and Q is a left divisor of R. Thus, Q is a right strict adjoint of P. □

Our main motivation in defining strict adjoints is the following readily provable

Corollary 2.12. *Let $P: \Lambda U \to \Lambda U$ be a nonsingular polynomial matrix, and let Q be a right (respectively left) strict adjoint of P. Also, let $\delta: \Lambda U \to \Lambda U$ be a diagonal polynomial matrix. If P is a polynomial left (respectively right) divisor of δ, then so is also PQ (respectively QP).*

Further, let $P: \Lambda U \to \Lambda U$ be a polynomial matrix. We say that P is *row reduced* if the greatest common divisor of the entries in each row of P is 1. In particular, by our construction in proof of Lemma 2.11, it follows that *every left strict adjoint is row reduced.* Row reduced matrices have the following useful property, which can be readily verified.

Lemma 2.13. *Let $P: \Lambda U \to \Lambda U$ be a nonsingular row reduced polynomial matrix, and let $\delta: \Lambda U \to \Lambda U$ be any diagonal matrix. If δP is input/output stable, then so is also δ.*

Finally, we shall frequently use the following type of diagonal matrices. Let $f: \Lambda U \to \Lambda Y$ be a rational matrix, and let $\delta: \Lambda Y \to \Lambda Y$ be a nonsingular diagonal matrix. We say that δ is a *left diagonal stabilizer* of f whenever the following hold: (i) δf is input/output stable, and (ii) if $\delta': \Lambda Y \to \Lambda Y$ is any diagonal matrix such that $\delta' f$ is input/output stable, then δ is a σ-divisor of δ'. A left diagonal stabilizer for f can be constructed as follows. For all $i = 1,\dots,p\ (=\dim Y)$, let f^i be the i-th row of f, and let (ψ_i) be the (polynomial) ideal of all nonzero elements $\alpha \in \Omega^+ K$ such that αf^i is input/output stable. Then, defining $\delta := \mathrm{diag}(\psi_1,\dots,\psi_p)$, we evidently have that δf is input/output stable. Moreover, a direct verification shows that δ is indeed a left stabilizer of f. Any other diagonal left stabilizer of f is the multiple of δ by a diagonal $\Omega_\sigma^+ K$-unimodular matrix. We thus obtain the following

Lemma 2.14. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then, f has a left diagonal stabilizer $\delta: \Lambda Y \to \Lambda Y$. Also, δ can be chosen polynomial.*

3. Decoupling

Let $f: \Lambda U \to \Lambda Y$ be a linear input/output map. We say that f is *decoupled* if its transfer matrix (relative to specified bases $u_1,\dots,u_m \in U$ and $y_1,\dots,y_p \in Y$) is a diagonal matrix. (Of course, the decoupling problem depends on the choice of the bases $u_1,\dots,u_m$ in U and $y_1,\dots,y_p$ in Y. We assume that these bases have been chosen in accordance with the required decoupling strategy, and we leave them fixed throughout discussion. All matrix representations are relative to these bases.) Further, f is called (*dynamic output*) *feedback decouplable* if there exists a causal ΛK-linear (feedback) map $r: \Lambda U \to \Lambda U$ such that f_r is both decoupled and internally stable. Our main objective in the present section is to derive a complete characterization of the class of all nonsingular linear input/output maps $f: \Lambda U \to \Lambda U$, which are feedback decouplable. We restrict our attention exclusively to square nonsingular matrices, even though some of the following results hold under more general situations as well. This will simplify both our notation and our discussion.

We start with an auxiliary result, which is related to the problem of "zero assignment" by dynamic output feedback. Let $A, B: \Lambda U \to \Lambda U$ be nonsingular polynomial matrices. We say that the multiple AB is *interchangeable* whenever there exist polynomial matrices $\tilde{A}, \tilde{B}: \Lambda U \to \Lambda U$ satisfying (i) $AB = \tilde{B}\tilde{A}$, (ii) $\tilde{B}$ is left coprime with A, and (iii) $\tilde{A}$ is right coprime with B. Whenever conditions (ii) and (iii) hold, we call the equation $AB = \tilde{B}\tilde{A}$ an *interchange equation*. Interchangeable matrices have found several applications in control theory. In this connection, Wolovich [1978] proved that polynomial matrices A and B are interchangeable if and only if there exist polynomial matrices Y_1, Y_2 such that $Y_2 A + BY_1 = I$. Motivated by the latter condition, interchangeable matrices are sometimes called *skew coprime*. We now have the following

Proposition 3.1. *Let $f: \Lambda U \to \Lambda U$ be a nonsingular linear input/output map with a canonical zero representation $f = PQ^{-1}$. Then the following hold.*

(i) *Let r be a causal feedback map such that f_r is internally stable. Let $f_r = \hat{P}\hat{Q}^{-1}$ be a canonical zero representation of f_r. Then there exists a polynomial matrix S such that $\hat{P} = PS$. Furthermore, if T is any polynomial left divisor of S then the multiple PT is interchangeable.*

(ii) *Let S be a completely unstable nonsingular polynomial matrix such that PS is interchangeable. Then there exists a causal feedback map r such that f_r is internally stable, and PS is a left divisor of $\hat{P}$ in a canonical zero representation $f_r = \hat{P}\hat{Q}^{-1}$ of f_r.*

Proof. (i) Let $r = D^{-1}N$ be a canonical stability representation of r. Since f_r is internally stable, it follows from Proposition 2.10 that $V := DQ + NP$ is an $\Omega_\sigma^+ K$-unimodular matrix. Now

$$\hat{P}\hat{Q}^{-1} = f_r = f(I + rf)^{-1} = PV^{-1}D$$

(where we note that $\hat{Q}$ is $\Omega_\sigma^+ K$-unimodular). The matrix $S := V^{-1}D\hat{Q} = P^{-1}\hat{P}$ is evidently input/output stable, and, since P is completely unstable, it follows from Lemma 2.4 that S is a polynomial matrix. Let T be a left divisor of S, and let T_1 be the polynomial matrix satisfying $S = TT_1$. Now define $X_1 := T_1\hat{Q}^{-1}Q$ and $X_2 := V^{-1}N$. Clearly, both of X_1 and X_2 are input/output stable. Furthermore,

$$\begin{aligned} I &= V^{-1}DQ + V^{-1}NP = S\hat{Q}^{-1}Q + X_2 P = TT_1\hat{Q}^{-1}Q + X_2 P \\ &= TX_1 + X_2 P. \end{aligned}$$

Let ψ be in σ such that ψX_1 and ψX_2 are both polynomial matrices. Since P is completely unstable, it follows that $\det P$ and ψ are coprime polynomials. Hence, there exist polynomials π, χ such that $\pi\psi + \chi \det P = 1$. We now have

$$I = \pi\psi I + \chi \det PI = \pi\psi T X_1 + \pi\psi X_2 P + (\chi \operatorname{adj} P)P$$
$$= T(\pi\psi X_1) + (\pi\psi X_2 + \chi \operatorname{adj} P)P.$$

Let $Y_1 := \pi\psi X_1$ and $Y_2 := \pi\psi X_2 + x \operatorname{adj} P$. Clearly, Y_1 and Y_2 are polynomial matrices and $TY_1 + Y_2 P = I$. By the result of Wolovich [1978] mentioned above, P and T are interchangeable.

(ii) Since P and S are interchangeable, there exist polynomial matrices $\tilde{S}$, $\tilde{P}$ such that $PS = \tilde{S}\tilde{P}$ is an interchange equation. Further, $\tilde{S}$, $\tilde{P}$ may be chosen such that $\tilde{S}^{-1}$ is a causal matrix. (For example, if $\tilde{S}$ is chosen such that its columns constitute a proper basis, then $\tilde{S}^{-1}$ is causal.) Now consider the strictly causal input/output map $f_1 := \tilde{S}^{-1} f = \tilde{S}^{-1} P Q^{-1} = \tilde{P} S^{-1} Q^{-1}$. We will now show that $\tilde{P}$ and QS are right σ-coprime. Let Y_1, Y_2, Y_3, and Y_4 be input/output stable matrices such that

$$Y_1 P + Y_2 Q = I, \qquad Y_3 \tilde{P} + Y_4 S = I.$$

We now have

$$I = Y_1 P + Y_2 Q = Y_1 \tilde{S}\tilde{S}^{-1} P + Y_2 Q = Y_1 \tilde{S}\tilde{P} S^{-1} + Y_2 Q.$$

Consequently, $S = Y_1 \tilde{S}\tilde{P} + Y_2 QS$. Multiplying this equation on the left by Y_4, it follows that

$$I = (Y_3 + Y_4 Y_1 \tilde{S})\tilde{P} + (Y_4 Y_2) QS.$$

This shows that $\tilde{P}$ and QS are right σ-coprime.

Let r_1 be causal feedback map such that $f_{1_{r_1}}$ is internally stable. (For the existence of r_1, see Brasch and Pearson [1970]). Let $r_1 = D_1^{-1} N_1$ be a canonical stability representation of r_1. It follows from Proposition 2.10 that

$$V_1 := D_1 QS + N_1 \tilde{P}$$

is a $\Omega_\sigma^+ K$-unimodular matrix. Define $r := r_1 \tilde{S}^{-1} = D_1^{-1} N_1 \tilde{S}^{-1}$. Clearly, r is causal, and we next show that f_r is internally stable, using 2.9. First,

$$l_r = (I + rf)^{-1} = (I + r_1 f_1)^{-1} = QSV_1^{-1} D_1$$

is input/output stable. Now $h := Q^{-1} l_r = SV_1^{-1} D_1$ is also input/output stable. Further, $hr = SV_1^{-1} N_1 \tilde{S}^{-1}$, and, upon showing that hr is also input/output stable, it will follow by Corollary 2.9 that f_r is internally stable. Evidently, $V_1^{-1} D_1 QS + V_1^{-1} N_1 \tilde{P} = I$, which in turn implies $S(V_1^{-1} D_1 Q + V_1^{-1} N_1 \tilde{P} S^{-1}) = I$. It follows that

$$hrP = SV_1^{-1} N_1 \tilde{S}^{-1} P = I - SV_1^{-1} D_1 Q$$

is input/output stable. But then, since $\tilde{S}$, P are left coprime, we conclude that

$hr = SV_1^{-1}N_1\tilde{S}^{-1}$ is also input/output stable. Thus, by Corollary 2.9 f_r is internally stable. Finally,

$$\hat{P}\hat{Q}^{-1} = f_r = f(I+rf)^{-1} = PSV_1^{-1}D_1.$$

Hence, $(PS)^{-1}\hat{P} = V_1^{-1}D_1\hat{Q}$ is input/output stable, and, since PS is a completely unstable polynomial matrix, it follows that PS is a left divisor of $\hat{P}$. This concludes the proof. □

The above proposition shows that interchangeability is a key condition in zero-assignment problems. There are many problems in control theory in which zero-assignment is the main issue. Most notable among these are output regulation and tracking problems. As expected, interchangeability plays a major role in these problems. (See Wolovich and Ferreira [1979] and the references cited there.)

It will be seen that the various constructions involved in the above proposition turn out to be quite important in the proof of our main theorem 3.4.

We now return to the examination of the decoupling problem. Let f be a nonsingular linear input/output map, and let $f = PQ^{-1}$ be its canonical zero representation. Suppose that there exists a causal feedback map r such f_r is internally stable and decoupled. Since f_r is decoupled, there exists a canonical zero representation $f_r = \hat{P}\hat{Q}^{-1}$, where both of $\hat{P}$ and $\hat{Q}$ are diagonal matrices. Such a representation will be called a *diagonal zero representation*. Now, by Proposition 3.1 (i), P is a left divisor of $\hat{P}$. Whence, since $\hat{P}$ is diagonal, it follows by Corollary (2.12) that PP_*, where P_* is a right strict adjoint of P, also is a left divisor of $\hat{P}$. Consequently, Proposition 3.1 (i) implies that P and P_* are interchangeable. We have proved the following

Corollary 3.2. *Let $f\colon \Lambda U \to \Lambda U$ be a nonsingular linear input/output map, and assume that f is feedback decoupleable. Let $f = PQ^{-1}$ be a canonical zero representation, and let P_* be a right strict adjoint of P. Then P and P_* are interchangeable.*

Thus, a necessary condition for feedback decoupling is that the zero matrix and its right strict adjoint be interchangeable. In the next proposition, we describe a certain class of polynomial matrices which are interchangeable with their right strict adjoints. It is interesting to note that this class is "dense" in the set of all square polynomial matrices.

Proposition 3.3. *Let $P\colon \Lambda U \to \Lambda U$ be a nonsingular polynomial matrix, and let P_* be its right strict adjoint. Also, let $\varepsilon_1,\dots,\varepsilon_m$ be the invariant factors of P, where ε_i divides ε_{i+1}, $i = 1,\dots,m-1$. If the polynomials ε_i and $\varepsilon_m/\varepsilon_i$ are coprime for all $i = 1,\dots,m-1$, then P and P_* are interchangeable.*

We note, in particular, that if the matrix P is *cyclic*, that is, if $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_{m-1} = 1$, then it follows by Proposition 3.3 that P and P_* are interchangeable.

Proof. By classical results in the theory of polynomial matrices (e.g., Macduffee [1934]), there exists a polynomial matrix $R\colon \Lambda U \to \Lambda U$ such that $PR = RP = \varepsilon_m I$. Also, by the Smith canonical form theorem, there exist polynomial unimodular matrices $M, N\colon \Lambda U \to \Lambda U$ such that $MPN = \operatorname{diag}(\varepsilon_1,\dots,\varepsilon_m) =: E$. Next, for all $i = 1,\dots,m$, since ε_i and $\varepsilon_m/\varepsilon_i$ are coprime, there exist polynomials α_i and β_i such that $\alpha_i\varepsilon_i^2 + \beta_i\varepsilon_m = \varepsilon_i$. Denoting $A := \mathrm{diag}(\alpha_1,\ldots,\alpha_m)$ and $B := \mathrm{diag}(\beta_1,\ldots,\beta_m)$, we obtain that $EAE + \varepsilon_m B = E$, or, equivalently $P(NAM)P + (PR)(M^{-1}BN^{-1}) = P$. Hence, $(NAM)P + R(M^{-1}BN^{-1}) = I$. Now, since PR is diagonal, there exists a polynomial matrix $C: \Lambda U \to \Lambda U$ such that $R = P_*C$. Consequently, $(NAM)P + P_*(CM^{-1}BN^{-1}) = I$, so that P and P_* are skew coprime, and thus, by Wolovich [1978], Theorem 1, they are interchangeable as well. □

We now turn to the problem of feedback decoupling. Let $A, B: \Lambda U \to \Lambda U$ be input/output stable matrices. We say that the (ordered) pair (A, B) is *d-coprime* if there exist input/output stable matrices $Y_1, Y_2: \Lambda U \to \Lambda U$, where Y_1 is *diagonal*, such that $Y_1A + Y_2B = I$. Explicit conditions for *d*-coprimeness are obtained in the Appendix. The following theorem is the main result of our present discussion. We shall discuss the explicit verification of its conditions after stating its proof.

Theorem 3.4. *Let $f: \Lambda U \to \Lambda U$ be a nonsingular linear input/output map, and let $f = PQ^{-1}$ be a canonical zero representation. Let P_* be a right strict adjoint of P. Then the following statements* (i), (ii), *and* (iii) *are equivalent.*

(i) *f is feedback decoupleable.*
(ii) (a) *$f^{-1}|(0)$ is a diagonal matrix, and*
 (b) *there exist polynomial matrices Y_3 and Y_4, where Y_3 is diagonal, such that $P_*Y_3Q + Y_4P = I$.*
(iii) (a) *$f^{-1}|(0)$ is a diagonal matrix, and*
 (b) *there exists an interchange equation $PP_* = \tilde{P}_*\tilde{P}$ such that $(QP_*, \tilde{P})$ is d-coprime.*

The requirement (ii) (a) that the polynomial part of f^{-1} be a diagonal matrix reflects the requirement that the decoupling feedback-compensator r be causal (compare to Wolovich [1975], and Bayoumi and Duffield [1977]). The further condition (ii) (b), or, equivalently, (iii) (b), guarantees the internal stability of the decoupled system. As we see, the strict adjoint P_* plays a dominant role in the conditions for decoupling.

Before proving Theorem 3.4, we need a preliminary instrumental discussion. Let $A: \Lambda U \to \Lambda U$ be a matrix, and let a_{ij} in ΛK, where $i, j = 1,2,\ldots,m$, be the entries of A. We denote $A_d := \mathrm{diag}(a_{11}, a_{22},\ldots,a_{mm})$ (the *diagonal part* of A), and $A_{\mathrm{off}} := A - A_d$ (the *off-diagonal part* of A). We will need the following

Lemma 3.5. *Let $f: \Lambda U \to \Lambda U$ be a nonsingular linear input/output map, and let $f = PQ^{-1}$ be a canonical zero representation of f. Let P_* be a right strict adjoint of P, and denote $A := QP_*$. If $f^{-1}|(0)$ is a diagonal matrix, then the diagonal part A_d is nonsingular.*

Proof. By Theorem 2.7, there exists a causal ΛK-linear map $r: \Lambda U \to \Lambda U$ such that $f_r = (f^{-1}|(0))^{-1}$. Assume now that $f^{-1}|(0)$ is diagonal, and let $f^{-1}|(0) = \mathrm{diag}(\alpha_1, \alpha_2,\ldots,\alpha_m)$. Then, since $f_r = fl_r$ and l_r is bicausal, f_r is nonsingular and strictly causal. Hence $\mathrm{ord}\,\alpha_i \le -1$ for all $i = 1,2,\ldots,m$. Also, $f_r^{-1} = l_r^{-1}f^{-1} = f^{-1} + r = QP^{-1} + r$, so that

$$QP_* + rPP_* = \mathrm{diag}(\alpha_1, \alpha_2,\ldots,\alpha_m)PP_*.$$

Since PP_* is diagonal, we obtain that

$$A_d = (\mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m) - r_d) PP_*,$$

where r_d is the diagonal part of r. But then, since r is causal, and since PP_* is nonsingular and $\mathrm{ord}\, \alpha_i \leq -1$ for all $i = 1, 2, \ldots, m$, A_d is nonsingular. □

The following is an auxilliary technical result.

Lemma 3.6. *Let $A, B, C: \Lambda U \to \Lambda U$ be input/output stable matrices. Assume that the pair (A, B) is d-coprime, that the diagonal part A_d is nonsingular, and that CB is diagonal and nonsingular. Then there exist input/output stable matrices Y_1 and Y_2, where Y_1 is diagonal nonsingular, such that $Y_1 A + Y_2 B = I$, and the diagonal entries of the matrix $Y_1^{-1} Y_2 C^{-1}$ are causal.*

Proof. We will need the following notation. Let X be a matrix, and let $\{x_{ij}\}$ be its entries. We denote $\underline{\mathrm{ord}}\, X := \min_{i,j}\{\mathrm{ord}\, x_{ij}\}$ and $\overline{\mathrm{ord}}\, X := \max_{i,j}\{\mathrm{ord}\, x_{ij}\}$. By the d-coprimeness of (A, B), there exist input/output stable matrices Y_3, Y_4 such that Y_3 is diagonal and $Y_3 A + Y_4 B = I$. Defining $D := Y_4 C^{-1}$, and considering diagonal parts, we obtain the diagonal equation (recall that CB is diagonal)

$$Y_3 A_d + D_d(CB) = I.$$

Now, we let β be any nonsingular, diagonal, and input/output stable matrix such that $\beta(CB) = \varphi I$ is scalar, and φ belongs to $\Omega_\sigma^+ K$. We also choose a polynomial ψ in the stability set σ with sufficiently high degree, such that

$$\mathrm{ord}\, \psi < -|\underline{\mathrm{ord}}\, \beta A_d| - |\underline{\mathrm{ord}}\, A_d| - |\underline{\mathrm{ord}}\, CB|. \tag{*}$$

Multiplying by ψ, we obtain $(\psi Y_3) A_d + (\psi D_d)(CB) = \psi I$. The following manipulation is intended to transfer the high-degree components of ψD_d to a product of A_d. We define the quantities $\delta := [\psi D_d(\beta A_d)^{-1}]|(0)$ (the polynomial part); $\alpha := D_d - \psi^{-1}\delta\beta A_d$; and $Y_1 := Y_3 + \psi^{-1}\delta\beta CB$, so that

$$Y_1 A_d + \alpha CB = I,$$

and Y_1 is diagonal and stable. Assume for a moment that Y_1 is nonsingular and that $Y_1^{-1}\alpha$ is causal. Then, since $\beta CB = \varphi I$ is scalar, we have $Y_1 A = (Y_3 + \psi^{-1}\delta\beta CB) A = Y_3 A + \psi^{-1}\delta A(\beta CB)$. Defining $Y_2 := Y_4 - \psi^{-1}\delta A \beta C$, we obtain that

$$Y_1 A + Y_2 B = I;$$

Y_2 is input/output stable; and $(Y_1^{-1} Y_2 C^{-1})_d = Y_1^{-1}(Y_2 C^{-1})_d = Y_1^{-1}\alpha$. Thus, our proof will conclude upon showing that Y_1 is nonsingular, and that $Y_1^{-1}\alpha$ is causal.

Now by definition, $(\psi\alpha)(\beta A_d)^{-1}$ is causal, so that $\underline{\mathrm{ord}}\, \psi\alpha \geq \underline{\mathrm{ord}}\, \beta A_d$. Further, for the same reason, $\underline{\mathrm{ord}}(\psi\alpha)(CB) = \underline{\mathrm{ord}}(\psi\alpha)(\beta A_d)^{-1}(\beta A_d CB) \geq \underline{\mathrm{ord}}(\beta A_d CB) \geq \underline{\mathrm{ord}}\, \beta A_d + \underline{\mathrm{ord}}\, CB$, so that by (*), $\overline{\mathrm{ord}}(\psi\alpha CB) > \mathrm{ord}\, \psi$. Whence, since $\psi Y_1 A_d + \psi\alpha CB = \psi I$, the leading coefficients of the columns of $\psi Y_1 A_d$ are the same as those in ψI. By diagonality, this implies that Y_1 is nonsingular, and that

$$\underline{\mathrm{ord}}\, \psi Y_1 A_d = \overline{\mathrm{ord}}\, \psi Y_1 A_d = \mathrm{ord}\, \psi.$$

Consequently, using (*), $\overline{\mathrm{ord}}\, \psi Y_1 = \mathrm{ord}\, \psi - \underline{\mathrm{ord}}\, A_d < -|\underline{\mathrm{ord}}\, \beta A_d| \le \underline{\mathrm{ord}}\, \beta A_d \le \underline{\mathrm{ord}}$ $\psi\alpha$, where the last step is by the present paragraph. Thus,

$$\overline{\mathrm{ord}}\, \psi Y_1 < \underline{\mathrm{ord}}\, \psi\alpha,$$

so that $(\psi Y_1)^{-1}\psi\alpha = Y_1^{-1}\alpha$ is causal, and our proof concludes. □

Proof of Theorem 3.4. We will prove the sequence (i) ⇒ (ii) ⇒ (iii) ⇒ (i) of implications. We start with (i) ⇒ (ii). Let f be feedback decoupleable. Let r be a causal feedback map such that f_r is both decoupled and internally stable. Then, since f_r is diagonal, so is also $(f_r^{-1})|(0)$. Hence, by Theorem 2.7, $f^{-1}|(0)$ is also diagonal. Let $r = D^{-1}N$ be a canonical stability representation of r, and let $f_r = \hat{P}\hat{Q}^{-1}$ be a diagonal zero representation of f_r. Now

$$\hat{P}\hat{Q}^{-1} = f_r = f(I + rf)^{-1} = PV^{-1}D,$$

where $V := DQ + NP$. Internal stability of f_r implies that V is an $\Omega_\sigma^+ K$-unimodular matrix. By Proposition 3.1(i), P is a left divisor of $\hat{P}$. Since $\hat{P}$ is a diagonal matrix, it follows that PP_* is also a left divisor of $\hat{P}$. Hence $\hat{P} = PP_*T$, for some diagonal polynomial matrix T. Define $Y_3 := T\hat{Q}^{-1}$ and $Y_4 := V^{-1}N$. Clearly, Y_3 and Y_4 are both input/output stable. Now

$$\begin{aligned} I &= V^{-1}DQ + V^{-1}NP = P^{-1}\hat{P}\hat{Q}^{-1}Q + V^{-1}NP = P_*(T\hat{Q}^{-1})Q + Y_4P \\ &= P_*Y_3Q + Y_4P. \end{aligned}$$

Since both of T and $\hat{Q}$ are diagonal, Y_3 is diagonal as well. This shows that (i) ⇒ (ii).

We will now show (ii) ⇒ (iii). By (ii), $f^{-1}|(0)$ is a diagonal matrix. Let Y_3 and Y_4 be input/output stable matrices such that Y_3 is diagonal and

$$P_*Y_3Q + Y_4P = I.$$

(So that, in particular, P_* and Y_4 are σ-coprime). This equation can be rewritten as

$$Y_3QP_* + P_*^{-1}Y_4PP_* = I.$$

Let $Y_2\tilde{P}_*^{-1}$ be a canonical pole representation of the (σ-coprime factorization) $P_*^{-1}Y_4$. Then $Y_3QP_* + Y_2\tilde{P}_*^{-1}PP_* = I$. Since Y_2 and $\tilde{P}_*$ are right σ-coprime and since $Y_2\tilde{P}_*^{-1}PP_*$ $(= I - Y_3QP_*)$ is clearly input/output stable, it follows that $\tilde{P}_*^{-1}PP_*$ is also input/output stable. Since $\tilde{P}_*$ is completely unstable, we conclude by Lemma 2.4 that $\tilde{P}_*^{-1}PP_*$ is a polynomial matrix. Let $\tilde{P} := \tilde{P}_*^{-1}PP_*$. Then

$PP_* = \tilde{P}_*\tilde{P}$. Further,

$$Y_3QP_* + Y_2\tilde{P} = I, \tag{*}$$

so that, recalling that Y_3 is diagonal, we obtain that $(QP_*, \tilde{P})$ is d-coprime. The proof of (ii) ⇒ (iii) will be complete if we show that $PP_* = \tilde{P}_*\tilde{P}$ is an interchange equation. Clearly, by (*), $\tilde{P}$ and P_* are right σ-coprime, and, since P_* is completely unstable, it follows that $\tilde{P}$ and P_* are right polynomially coprime as well. Now $P_*^{-1}Y_4 = Y_2\tilde{P}_*^{-1}$ are both σ-coprime representations, and P_* and $\tilde{P}_*$ are both completely unstable polynomial matrices. It follows that $\det P_* = k \det \tilde{P}_*$, for some k in the field K. But then, since

$$\tilde{P}_*^{-1}P = \tilde{P}P_*^{-1}$$

and $\tilde{P}, P_*$ are right polynomial coprime, it follows that $\tilde{P}_*$ and P are left coprime. This completes the proof of (iii) ⇒ (ii).

The last step of this proof is to show that (iii) ⇒ (i). Assume that (iii) holds. Let $PP_* = \tilde{P}_*\tilde{P}$ be an interchange equation, and let Y_1, Y_2 be input/output stable matrices, where Y_1 is diagonal, such that $Y_1QP_* + Y_2\tilde{P} = I$. In view of Lemmas 3.5 and 3.6 (with $A = QP_*$, $B = \tilde{P}$, $C = \tilde{P}_*$), we can assume that Y_1^{-1} exists and that the diagonal entries of $Y_1^{-1}Y_2\tilde{P}_*^{-1}$ are causal. Define $r := Y_1^{-1}Y_2\tilde{P}_*^{-1}$. We claim that r decouples f with internal stability. We have $rf = Y_1^{-1}Y_2\tilde{P}_*^{-1}PQ^{-1} = Y_1^{-1}Y_2\tilde{P}P_*^{-1}Q^{-1}$. It now follows that

$$I + rf = I + Y_1^{-1}Y_2\tilde{P}P_*^{-1}Q^{-1} = Y_1^{-1}P_*^{-1}Q^{-1}.$$

Hence

$$f_r = f(I + rf)^{-1} = PQ^{-1}QP_*Y_1 = PP_*Y_1$$

is diagonal.

We now show that r is causal and f_r is internally stable. Since the diagonal entries of $Y_1^{-1}Y_2\tilde{P}_*^{-1}$ are causal, the diagonal part r_d of r is causal. Further, as $QP_* + rPP_* = Y_1^{-1}$, we have that $f_r^{-1} = f^{-1} + r = Y_1^{-1}(PP_*)^{-1}$. Since the right hand side is diagonal,

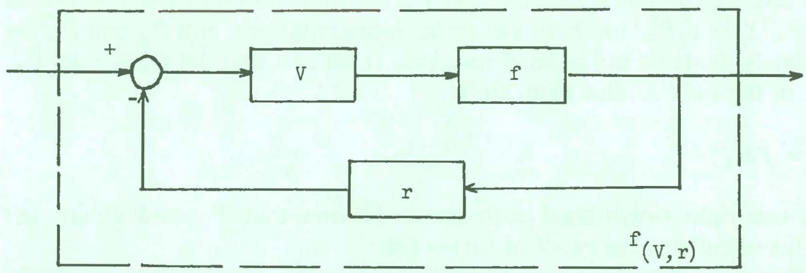
$$r_{\text{off}} = -(f^{-1})_{\text{off}}.$$

But then, since $f^{-1}|(0)$ is diagonal, it follows that r_{off} is causal. Thus, $r\,(= r_d + r_{\text{off}})$ is causal. To prove internal stability, we use (2.9). First, $h := Q^{-1}l_r = Q^{-1}(I + rf)^{-1} = P_*Y_1$, is clearly input/output stable. Further, $hr = P_*Y_2\tilde{P}_*^{-1}$. We now have

$$hrP = P_*Y_2\tilde{P}_*^{-1}P = P_*Y_2\tilde{P}P_*^{-1} = I - P_*Y_1Q.$$

Thus, hrP is input/output stable. Since $\tilde{P}_*$ and P are left coprime, it follows that hr is input/output stable as well. We can now conclude from Corollary 2.9 that f_r is internally stable. This completes the proof of the theorem. □

Before turning to an examination of Theorem 3.4, we show that this theorem allows the solution of a somewhat more general problem than the one we started with. In particular, we show that, by a simple modification, one can obtain a solution to the problem of decoupling through a combination of static (constant gain) precompensation and dynamic output feedback. To this end, consider the following diagram



where $f: \Lambda U \to \Lambda Y$ is a linear input/output map, $V: U \to U$ is a static nonsingular precompensator, and $r: \Lambda Y \to \Lambda U$ is a causal ΛK-linear map. We denote by $f_{(V,r)}$ the resulting system. Then, we have that

$$f_{(V,r)} = (fV)[I + r(fV)]^{-1},$$

and it follows that there exists a pair (V, r) such that $f_{(V,r)}$ is decoupled if and only if there exists a nonsingular $V: U \to U$ such that (fV) is feedback decoupleable.

Assume next that $f: \Lambda U \to \Lambda U$ is nonsingular. Then, since V is static, we have that $(fV)^{-1}|(0) = V^{-1}[f^{-1}|(0)]$. Now, by Theorem 3.4 (i), a necessary condition for decoupling is that $V^{-1}[f^{-1}|(0)]$ be diagonal. Moreover, by nonsingularity, the last condition determines V up to a diagonal static right multiplier, which has no effect on conditions (ii) and (iii) of Theorem 3.4. Thus, we obtain the following

Corollary 3.7. *Let $f: \Lambda U \to \Lambda U$ be a nonsingular linear input/output map. Then, there exist a nonsingular static precompensator $V: U \to U$ and a causal output feedback $r: \Lambda U \to \Lambda U$ such that $f_{(V,r)}$ is both decoupled and internally stable if and only if the following hold*: (i) *There exists a nonsingular $V: U \to U$ such that $V^{-1}[f^{-1}|(0)]$ is a diagonal matrix.* (ii) *(fV) is feedback decoupleable.*

We remark that condition (i) in the above Corollary can be easily checked as follows. For all $i = 1, \ldots, m$, let ψ_i be the (polynomial) greatest common divisor of all entries in column i of $f^{-1}|(0)$, and let $\Psi := \text{diag}(\psi_1, \ldots, \psi_m)$. Then, condition (i) is satisfied if and only if $[f^{-1}|(0)]\Psi^{-1}$ is a static matrix (in which case it can be taken as V).

Using the above observation, all our following discussion can be generalized to include static precompensation as well.

We consider now a simple particular case of Theorem 3.4. Let $f: \Lambda U \to \Lambda U$ be a linear input/output map. We say that f is *σ-invertible* if f^{-1} exists and is input/output stable. Intuitively speaking, f is σ-invertible whenever it (is nonsingular and) has no unstable zeros (For a detailed discussion of σ-invertible maps, see Hammer [1983a].) Assume now that $f: \Lambda U \to \Lambda U$ is a σ-invertible linear input/output map. Then, f possesses a canonical zero representation $f = PQ^{-1}$, with $P = I$. Consequently, in the notation of Theorem 3.4, we also have $P_* = I$ and $\tilde{P} = I$. But then, condition (ii)b of the theorem is evidently satisfied, and we obtain the following

Corollary 3.8. *Let $f: \Lambda U \to \Lambda U$ be a σ-invertible linear input/output map. Then, f is feedback decoupleable if and only if $f^{-1}|(0)$ is a diagonal matrix.*

We next turn to an examination of the d-coprimeness condition. We start with a special case. (Here, recall that the concept of diagonal stabilizers was defined in Section 2.)

Lemma 3.9. *Let A, B be input/output stable matrices, where B is diagonal and nonsingular. Let C be a left diagonal stabilizer of $A_{\text{off}}B^{-1}$. Then, the pair (A, B) is d-coprime if and only if the diagonal matrices CA_d and B are σ-coprime.*

Proof. Suppose (A, B) is d-coprime. Then there exist input/output stable matrices Y_1, Y_2 such that Y_1 is diagonal and $Y_1A + Y_2B = I$. Considering diagonal and off-diagonal parts, we obtain

$$Y_1 A_{\text{off}} + Y_{2\text{off}} B = 0, \qquad Y_1 A_d + Y_{2d} B = I.$$

Now, since $Y_{2\text{off}} = -Y_1 A_{\text{off}} B^{-1}$ is input/output stable, C is a σ-divisor of Y_1. Hence, $Y_1 = Y_3 C$, where Y_3 is diagonal and input/output stable. But then, we obtain the diagonal equation

$$Y_3(CA_d) + Y_{2d}B = I,$$

so that CA_d, B are σ-coprime.

Conversely, assume that CA_d and B are σ-coprime. Then there exist diagonal input/output stable matrices Y_4, Y_5 such that $Y_4CA_d + Y_5B = I$. Define $Y_1 := Y_4C$, $Y_2 = Y_5 - Y_1A_{\text{off}}B^{-1}$. We then obtain that Y_2 is input/output stable and $Y_1A + Y_2B = I$. Thus, (A, B) is d-coprime. □

As a consequence of Lemma 3.9, we obtain another interesting particular case of the decoupling problem. Let f be a nonsingular linear input/output map. We say that f is *zero decoupled* if there exists a canonical zero representation $f = PQ^{-1}$ with P a diagonal matrix. Given a nonsingular linear input/output map f, it is quite easy to check whether f is zero decoupled, using the following procedure. Let $f = PQ^{-1}$ be a canonical zero representation of f. For each $i = 1, 2, \ldots, m$, let μ_i be the greatest common (polynomial) divisor of the entries in the i-th row of P. Let $\mu := \text{diag}(\mu_1, \mu_2, \ldots, \mu_m)$. Then it is readily seen that f is zero decoupled if and only if $\mu^{-1}P$ is a unimodular polynomial matrix, i.e., if and only if $\det(\mu^{-1}P)$ belongs to the field K.

Let f be a zero decoupled input/output map. Let $f = PQ^{-1}$ be a canonical zero representation such that P is diagonal. In the notation of Theorem 3.4, we can choose $P_* = I$, $\tilde{P} = P$. Using Theorem 3.4 and Lemma 3.9 we obtain the following

Corollary 3.10. *Let $f: \Lambda U \rightarrow \Lambda U$ be a zero decoupled nonsingular input/output map, and let $f = PQ^{-1}$ be a canonical zero representation such that P is a diagonal matrix. Also, let R be a left diagonal stabilizer of the off diagonal part $(f^{-1})_{\text{off}}$. Then f is feedback decoupleable if and only if the following conditions hold:*

(i) *$f^{-1}|(0)$ is a diagonal matrix.*

(ii) *The diagonal matrices RQ_d and P are σ-coprime.*

Remark 3.11. Let P, Q, P_* be as in the statement of Theorem 3.4. Suppose P and P_* are interchangeable. Let $PP_* = \tilde{P}_*\tilde{P}$ be an interchange equation. It is not difficult to show that QP_* and $\tilde{P}$ are right σ-coprime, i.e., there exist input/output stable matrices Y_1 and Y_2 such that $Y_1QP_* + Y_2\tilde{P} = I$. (This is essentially shown in the proof of Proposition 3.1.) For decoupling, however, we need an additional condition, i.e., that Y_1 be diagonal. Thus, d-coprimeness plays a central role in decoupling problems.

Example 3.12. We will now illustrate our results with an example. Consider the stability set $\sigma := \{f \text{ in } \Omega\mathbf{R} : f(z) = 0 \text{ implies } \operatorname{Re}(z) < 0\}$. Let $f := PQ^{-1}$, where

$$P = \begin{bmatrix} z^2 - 1 & z^2 + z - 2 \\ z^2 - 2z & z^2 - z - 2 \end{bmatrix}, \qquad Q = \begin{bmatrix} z^3 + 4z^2 & z^3 + 5z^2 + 3z \\ z^3 + 3z^2 & z^3 + 4z^2 + 6z \end{bmatrix}$$

As $\det P = z^3 - 3z + 2$ and $\det Q = (z^2 - 9)z^2$, it follows that PQ^{-1} is a σ-coprime factorization of f. Also,

$$P^{-1} = \begin{bmatrix} \dfrac{z+1}{z-1} & \dfrac{-(z+2)}{z-2} \\ \dfrac{-z}{z-1} & \dfrac{z+1}{z-2} \end{bmatrix}.$$

Using the construction given in the proof of Lemma 2.11, the right strict adjoint P_* of P is given by

$$P_* = \begin{bmatrix} z+1 & -(z+2) \\ -z & z+1 \end{bmatrix}.$$

Since P_* is unimodular, by defining $\hat{P} := PP_*$ and $\hat{Q} := QP_*$, we obtain the factorization $f = \hat{P}\hat{Q}^{-1}$, which is σ-coprime. Further,

$$\hat{P} = \begin{bmatrix} z-1 & 0 \\ 0 & z-2 \end{bmatrix}, \qquad \hat{Q} = \begin{bmatrix} z^2 & 3z \\ 3z & z^2 \end{bmatrix}.$$

Note that f is zero-decoupled. Thus, we can apply Corollary 3.10 to check if f is feedback decoupleable. In order to apply Corollary 3.10, we have to compute a left diagonal stabilizer of $(f^{-1})_{\mathrm{off}}$. An easy calculation gives

$$(f^{-1})_{\mathrm{off}} = \begin{bmatrix} 0 & \dfrac{3z}{z-2} \\ \dfrac{3z}{z-1} & 0 \end{bmatrix}.$$

By definition of left diagonal stabilizer (given in Section 2), the left diagonal stabilizer of $(f^{-1})_{\mathrm{off}}$ is

$$\alpha_0 = \begin{bmatrix} z-2 & 0 \\ 0 & z-1 \end{bmatrix}.$$

We will now check the decoupleability conditions given in Corollary 3.10. The strict polynomial part of f^{-1} is

$$f^{-1}|(0) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},$$

which is diagonal. Further,

$$\alpha_0 \hat{Q}_d = \begin{bmatrix} z^2(z-2) & 0 \\ 0 & z^2(z-1) \end{bmatrix}, \qquad \hat{P} = \begin{bmatrix} z-1 & 0 \\ 0 & z-2 \end{bmatrix}$$

are σ-coprime. Hence, f is feedback decoupleable. In order to find a decoupling feedback compensator, we will use the constructions of Theorem 3.4. For this we have to find Y_3 and Y_4 such that $Y_3\hat{Q} + Y_4\hat{P} = I$, Y_3 is diagonal, and $Y_3^{-1}Y_4$ is causal. Some straightforward calculations give

$$Y_3 = \begin{bmatrix} \dfrac{(z-17)(z-2)}{(z+1)^4} & 0 \\ 0 & \dfrac{(z-1)(4z+73)}{4(z+1)^4} \end{bmatrix},$$

$$Y_4 = \begin{bmatrix} \dfrac{23z^2-5z-1}{(z+1)^4} & \dfrac{-3z(z-17)}{(z+1)^4} \\ \dfrac{-3z(4z+73)}{4(z+1)^4} & \dfrac{-(53z^2+9z+2)}{4(z+1)^4} \end{bmatrix}.$$

Using the construction given the proof of Theorem 3.4, the feedback compensator

$r := Y_3^{-1}Y_4$ decouples f with internal stability. Finally, the decoupled system is

$$f_r = \begin{bmatrix} \dfrac{(z-1)(z-17)(z-2)}{(z+1)^4} & 0 \\ 0 & \dfrac{(z-2)(z-1)(4z+73)}{4(z+1)^4} \end{bmatrix}.$$

Appendix

In this appendix we will examine the various coprimeness-type conditions which arise in the statement of our main Theorem 3.4. We will give an explicit characterization of d-coprimeness, and will show that d-coprimeness can be essentially reduced to a coprimeness condition between diagonal matrices. We will also briefly describe certain procedures which may be useful in checking condition (iii) of Theorem 3.4.

Let us start with an examination of the d-coprimeness condition. We need the following notion. Let $A, B, \gamma : \Lambda U \to \Lambda U$, where γ is diagonal and nonsingular, be input/output stable matrices. We say that the (ordered) pair (A, B) is *row compatible modulo* γ if there exist input/output stable matrices $\delta, S : \Lambda U \to \Lambda U$, where δ is diagonal, such that $A = \delta B + \gamma S$. We next show that row compatibility is essentially equivalent to coprimeness of certain diagonal matrices. First, we establish our notation. For a matrix X, we denote by X^i the i-th row, and by $X_{i,j}$ the (i, j) entry. Now, let J denote the set of all $i \in 1,\dots,m$ for which $B^i \neq 0$, and let (the difference set) $J^c := \{1,\dots,m\} \setminus J$. For all $i \in J$, let β_i be the greatest common σ-divisor of the entries in B^i, and let $\psi_1^i,\dots,\psi_m^i \in \Omega_\sigma^+ K$ be elements such that $\sum_{j=1}^{m} B_{i,j}\psi_j^i = \beta_i$. Further, for all $i \in J^c$, let $\beta_i := 1$, and let $\beta := \mathrm{diag}(\beta_1,\dots,\beta_m)$. Finally, denote $\alpha_i := \sum_{j=1}^{m} A_{i,j}\psi_j^i$ for all $i \in J$, $\alpha_i = 0$ for all $i \in J^c$, and $\alpha := \mathrm{diag}(\alpha_1,\dots,\alpha_m)$. Then, we have the following

Lemma A.1. *Let $A, B, \gamma : \Lambda U \to \Lambda U$, where γ is diagonal and nonsingular, be input/output stable matrices, and denote $\gamma := \mathrm{diag}(\gamma_1,\dots,\gamma_m)$. Then, in the above notation, the pair (A, B) is row compatible modulo γ if and only if the following hold.*

(i) *$\gamma^{-1}(A - \alpha\beta^{-1}B)$ is input/output stable.*

(ii) *The greatest common σ-divisor of γ_i and β_i is a σ-divisor of α_i for all $i = 1,\dots,m$.*

Proof. Assume first that the pair (A, B) is row compatible modulo γ. Then, there exist input/output stable matrices δ, S where δ is a diagonal matrix $\delta := \mathrm{diag}(\delta_1,\dots,\delta_m) : \Lambda U \to \Lambda U$, such that $A = \delta B + \gamma S$. Evidently, we can assume that $\delta_i = 0$ for all $i \in J^c$. Denoting $B^0 := \beta^{-1}B$, we have that B^0 is input/output stable, and $A=\delta\beta B^0+\gamma S$. Now, for all $i\in J$, we have $\sum_{j=1}^{m} B^0_{i,j}\psi^j_i=1$, so that $\alpha_i=\delta_i\beta_i+\gamma_i s_i$, where $s_i:=\sum_{j=1}^{m} S_{i,j}\psi^i_j$ is input/output stable. Hence, it follows that (ii) is necessary. Further, letting $s_i=0$ for all $i\in J^c$, and denoting $\Phi:=\operatorname{diag}(s_1,\dots,s_m)$, we obtain that $\alpha=\delta\beta+\gamma\Phi$, so that $\gamma^{-1}(A-\alpha\beta^{-1}B)=\gamma^{-1}[\delta\beta B^0+\gamma S-(\delta\beta+\gamma\Phi)B^0]=S-\Phi B^0$ is input/output stable, and (i) follows. Thus, both (i) and (ii) are necessary.

Conversely, assume that (i) and (ii) hold. Then, by (i), we have that $A=\alpha B^0+\gamma S'$, where S' is input/output stable. Next, by (ii), there exist input/output stable diagonal matrices $\delta,\Phi:\Lambda U\to\Lambda U$ such that $\alpha=\delta\beta+\Phi\gamma$. Consequently, $A=\delta\beta B^0+\gamma\Phi B^0+\gamma S'=\delta B+\gamma S$, where $S:=\Phi B^0+S'$ is input/output stable. Thus, the pair (A,B) is row compatible modulo γ, and our proof concludes. □

We next show that the condition of d-coprimeness is essentially a row compatibility condition, and thus, by Lemma A.1, it reduces to a coprimeness verification for diagonal matrices. To this end, let $A,B:\Lambda U\to\Lambda U$ be input/output stable matrices, and assume that B is polynomial and nonsingular. Then, by definition, the pair (A,B) is d-coprime if and only if there exist input/output stable matrices $Y_1,Y_2:\Lambda U\to\Lambda U$, where Y_1 is diagonal, such that $Y_1A+Y_2B=I$. Now, let $\operatorname{adj}B$ be the adjoint of B, i.e. the polynomial matrix consisting of the minors of B. Then, $B(\operatorname{adj}B)=\det B$, and, right multiplying the d-coprimeness condition by $\operatorname{adj}B$, we obtain the equivalent condition

$$\operatorname{adj}B = Y_1[A(\operatorname{adj}B)]+(\det B)Y_2$$

where we commuted the scalar $\det B$ and the matrix Y_2. Consequently, the pair (A,B) is d-coprime if and only if the pair $(\operatorname{adj}B, A(\operatorname{adj}B))$ is row compatible modulo $(\det B)I$. We restate this fact as the following

Proposition A.2. *Let $A,B:\Lambda U\to\Lambda U$ be input/output stable matrices, where B is polynomial and nonsingular. Then, the pair (A,B) is d-coprime if and only if the pair* $(\operatorname{adj}B, A(\operatorname{adj}B))$ *is row compatible modulo* $(\det B)I$.

Let us examine the linear equation arising in Theorem 3.4(iii). Given matrices P, Q, and P_*, we need to find input/output stable matrices Y_3 and Y_4, where Y_3 is diagonal, such that

$$P_*Y_3Q+Y_4P = I. \tag{A.3}$$

Using the Kronecker product of matrices (see Bellman [1970]), equation (A.3) can be transformed into a linear equation of the form $Ay=b$ (over the ring $\Omega^+_\sigma K$), where the matrix A and the vector b are given and have their entries in the ring $\Omega^+_\sigma K$, and where a solution vector y with entries in $\Omega^+_\sigma K$ is sought. This linear equation can be obtained as follows. Let $R_1,R_2,\dots,R_n$ denote the columns of an $n\times n$ matrix R. Let $C(R)$ be the "stacking operator" (see Bellman [1970, p. 245]), which transforms R into the column vector

$$C(R) := \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}.$$

Now by applying this stacking operator to (A.3), we obtain

$$(Q' \otimes P_*)C(Y_3) + (P' \otimes I)C(Y_4) = C(I),$$

where "$\otimes$" denotes the Kronecker product of two matrices, and "$'$" denotes the transpose. The restriction that Y_3 be diagonal is clearly equivalent to the requirement that certain entries of $C(Y_3)$ be zero. Let A_1 be the submatrix of $Q' \otimes P_*$ consisting of the 1st, the $(n+2)$nd, the $(2n+3)$rd,..., and the n^2th columns. Let $d(Y_3)$ be the vector obtained by writing the diagonal entries of Y_3 as a column vector. Then, (A.3) can be rewritten as

$$[(A_1 : P' \otimes I)] \begin{bmatrix} d(Y_3) \\ C(Y_4) \end{bmatrix} = c(I),$$

which is a linear equation of the form $Ay = b$ over the ring $\Omega_\sigma^+ K$. Since $\Omega_\sigma^+ K$ is a principal ideal domain, standard procedures based on the Hermite normal form may now be employed to find $d(Y_3)$ and $C(Y_4)$, and, whence, Y_3 and Y_4.

References

1. G. Basile, and G. Marro, A state-space approach to noninteracting controls, *Richerche di Automatica*, 1, 68–77 (1970).
2. M. M. Bayoumi, and T. L. Duffield, Output feedback decoupling and pole placement in linear time invariant systems, *Trans. IEEE*, AC-22, 143–142 (1977).
3. R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, (1970).
4. R. M. Brasch, and J. B. Pearson, Pole placement using dynamic compensators, *Trans. IEEE*, AC-15, 34–43 (1970).
5. C. A. Desoer, and W. S. Chan, The feedback interconnection of lumped linear time invariant systems, *J. Franklin Inst.*, 300, 335–351 (1975).
6. P. Falb, and W. Wolovich, Decoupling in the design of multivariable control systems, *IEEE Transactions on Automatic Control*, AC-12, 651–659 (1967).
7. E. G. Gilbert, The decoupling of multivariable systems by state feedback, *SIAM Journal on Control*, 7, 50–63 (1969).
8. I. C. Gokhberg, and M. G. Krein, Systems of integral equations, on a half line with kernels depending on the difference of arguments, *American Mathematical Society*: *Translations*, 14, 2:217–287 (1960).
9. J. Hammer, Stability and stable precompensation: An algebraic approach, *Math. Systems Theory* 16, 265–296 (1983).
10. J. Hammer, Linear dynamic output feedback: Invariants and stability, *IEEE Trans. on Automatic Control*, April 1983(b).

11. M. L. J. Hautus and M. Heymann, Linear feedback—an algebraic approach, *SIAM Journal on Control and Optimization*, 16, 83–105 (1978).
12. M. L. J. Hautus and M. Heymann, Linear feedback decoupling—transfer function analysis, Preprint, Department of Electrical Engineering, Technion, Israel Institute of Technology, Haifa, Israel (1980).
13. J. H. Howze, Necessary and sufficient conditions for decoupling using output feedback, *IEEE Transactions on Automatic Control*, AC-18, 44–46 (1973).
14. J. H. Howze, and J. B. Pearson, Decoupling and arbitrary pole-placement in linear systems using output feedback, *IEEE Transactions on Automatic Control*, AC-15, 660–663 (1970).
15. C. C. Macduffee, *The Theory of Matrices*, Chelsea, New York (1934).
16. B. S. Morgan, The synthesis of linear multivariable systems by state variable feedback, *IEEE Transactions on Automatic Control*, AC-9, 405–411 (1964).
17. A. S. Morse, System invariants under feedback and cascade control, *Lecture notes in Mathematical Systems*, Vol. 131 (G. Marchesini and S. Mitter, editors) Springer Verlag, Berlin (1975).
18. A. S. Morse, and W. M. Wonham, Decoupling and pole assignment by dynamic compensation, *SIAM Journal on Control*, 8, 317–337 (1970).
19. L. Pernebo, An algebraic theory for the design of controllers for linear multivariable systems–Part I: Structure matrices and feed-forward design, *Trans. IEEE*, AC-26, 171–183 (1981).
20. Z. V. Rekasius, Decoupling of multivariable systems by means of state variable feedback, Proceedings of the Third Annual Allerton Conference, pp. 439–448 (1965).
21. L. M. Silverman, Decoupling with state feedback and precompensation, *IEEE Transactions on Automatic Control*, AC-15, 487–489 (1970).
22. L. M. Silverman, and H. J. Payne, Input-output structure of linear systems with application to the decoupling problem, *SIAM Journal on Control and Optimization*, 9, 199–233 (1971).
23. S. H. Wang, and E. J. Davison, Design of decoupled systems: a frequency domain approach, *International Journal of Control*, 21, 529–536 (1975).
24. W. A. Wolovich, Output feedback decoupling, *IEEE Transactions on Automatic Control*, AC-20, 148–149 (1975).
25. W. A. Wolovich, Skew prime polynomial matrices, *IEEE Transactions on Automatic Control*, AC-23, 880–887 (1978).
26. W. A. Wolovich, and P. Ferreira, Output regulation and tracking in linear multivariable systems, *IEEE Transactions on Automatic Control*, AC-24, 460–465 (1979).
27. W. M. Wonham, *Linear Multivariable Control*, Springer, Berlin (1974).
28. W. M. Wonham, and A. S. Morse, Decoupling and pole-assignment in linear multivariable systems: a geometric approach, *SIAM Journal on Control*, 8, 1–18, (1970).
29. B. F. Wyman, *Linear systems over commutative rings*, Lecture notes, Stanford University, Stanford, California, U.S.A. (1972).
30. D. C. Youla, On the factorization of rational matrices, *IRE Trans. on Information Theory*, pp. 172–189 (1961).

Received August 31, 1983, and in final form on April 5, 1983.