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Control During Feedback Failure: Characteristics of the Optimal Solution

Debraj Chakraborty
Department of Electrical Engineering
Indian Institute of Technology Bombay
Powai, Mumbai-400076, India

Jacob Hammer
Department of Electrical and Computer Engineering
University of Florida
Gainesville, FL 32611-6130, USA

***Abstract*—The problem of keeping performance errors within bounds while controlling a perturbed open loop linear system is considered. The objective is to maximize the time during which performance errors remain acceptable, given that the controlled system is within a specified neighborhood of its nominal parameter values. It is shown that the optimal solution is associated with a switching function $z(t)$ which has the following feature: the optimal input signal is a bang-bang signal when $z(t)$ is not the zero function.**

I. INTRODUCTION

Feedback is often used to reduce the adverse effects of perturbations and uncertainties on the performance of control systems. However, feedback data may become unavailable due to disruptions or failures in feedback components or communication links. Additionally, economic or convenience considerations may dictate a policy whereby a feedback channel is opened only occasionally, when performance degrades below an acceptable level. It is therefore necessary to develop open loop controllers that maximize the duration of time during which a perturbed system can operate without feedback and not exceed acceptable error bounds. Examples of common applications where such controllers can be of benefit include a wide range of biomedical applications and networked control systems. For instance, in medical chemotherapy and in diabetes it is beneficial to maximize the time between observations and treatments to increase patient comfort and independence (e.g., [11]). In networked control systems, the need to reduce network traffic limits feedback to intermittent use (e.g., [7], [8], [14] and others). These and other applications motivate our efforts to develop open loop controllers that maintain low error operation over extended periods of time.

To introduce the problem in formal terms, consider a system Σ whose parameters are not known accurately. Denote by Σ_0 the nominal version of Σ, and let Σ_ε be the system obtained when the parameters of Σ are perturbed by ε. Here, the exact value of ε is not known; it is only known that ε does not exceed a specified bound d. For an input function $v(t)$, denote by $\Sigma_0 v$ the response of the nominal system, and let $\Sigma_\varepsilon v$ be the response of the perturbed system. The deviation of the response caused by the perturbation is then $|\Sigma_\varepsilon v - \Sigma_0 v|$. To reduce the impact of the perturbation, we design a controller that adds a "correction signal" $u(t)$ to the input signal v, so that the response of the perturbed system becomes $\Sigma_\varepsilon(v+u)$. The deviation between the perturbed and nominal output signals is then $|\Sigma_\varepsilon(v+u) - \Sigma_0 v|$. As the perturbation ε is not known, the correction signal $u(t)$ must be independent of ε. Now, let M be the maximal deviation allowed for the response, and let t_f be the duration of time during which

$$|\Sigma_\varepsilon(v+u)(t) - \Sigma_0 v(t)| \leq M. \tag{1}$$

Our objective is to find a correction signal $u(t)$ that maximizes the duration t_f, given that the perturbation ε is bounded by d. In line with physical reality, we assume that there is a bound K on the maximal input amplitude of the system Σ.

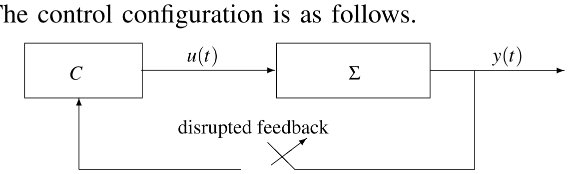
To simplify calculations, we concentrate in this note on the case where the nominal input signal $v(t)$ is the zero signal and where the nominal system Σ_0 has zero initial conditions, so that $\Sigma_0 v = 0$. Assuming that feedback is disconnected at the time $t = 0$, and letting x_0 be the state of the perturbed system at that time, inequality (1) becomes

$$|\Sigma_\varepsilon u(t)| \leq M \text{ for all } |\varepsilon| \leq d \text{ and all } 0 \leq t \leq t_f, \tag{2}$$

where $|u(t)| \leq K$ for all t. Our objective can then be described by the following.

Problem 1. Find a correction signal $u(t)$ that maintains (2) for the longest time t_f.

The control configuration is as follows.



Here, the controller C generates the corrective signal $u(t)$. The present note characterizes the optimal corrective signal $u(t)$, paying particular attention to conditions under which the optimal signal is a bang-bang function. Recall that a bang-bang function is a signal that switches between its extreme values; in our case, when $u(t)$ is a bang-bang signal, its components switch between the values K or $-K$. Needless to say, a bang-bang signal is characterized by its switching times, i.e., by the times at which its components switch from one extreme value to another. As a result, bang-bang signals are relatively easy to calculate and implement.

The main results of this note are in section III, where we show that the optimal input function $u(t)$ is associated with another function $z(t)$ that characterizes its features in the following way. The optimal input function $u(t)$ is a bang-bang function over all intervals of time over which $z(t)$ is not identically zero. Over such time intervals, $z(t)$ serves as a switching function: it determines the times at which the components of the optimal signal $u(t)$ change sign as they switch from one extreme value to the opposite one. On intervals over which the function $z(t)$ is identically zero, it provides no information about the optimal input signal $u(t)$. Nevertheless, we show in [2] that, over such intervals, the optimal input signal $u(t)$ can be approximated by a bang-bang function. Thus, bang-bang signals underlie our optimization problem.

Section II outlines the notation and framework of our discussion and indicates that our optimization problem is a max-min problem. As a result, the mathematical considerations rely on the large body of literature available in the area max-min optimization, including [5], [6], [9], [10], [12] and [13], the references cited in these works, and others.

II. Notation and Problem Formulation

Consider a linear time-invariant continuous-time system Σ described by a realization of the form

$$\Sigma : \dot{x}(t) = A'x(t) + B'u(t),\ x(0) = x_0. \tag{3}$$

Here, $x(t)$ is the state of Σ at the time t and $u(t)$ is the input function at the time t. Denote by n the dimension of $x(t)$ and by m the dimension of $u(t)$. Then, A' and B' are constant real matrices of dimensions $n \times n$ and $n \times m$, respectively. The initial state x_0 is the state of Σ at the time feedback was lost and thus is known.

Perturbations of the system Σ are represented by uncertainties about the entries of the matrices A' and B'. To describe these uncertainties, let $d > 0$ be a real number. Denote by Δ_A the set of all $n \times n$ matrices whose entries are in the interval $[-d, d]$, and let Δ_B be the set of all $n \times m$ matrices with entries in $[-d, d]$. Then, set

$$A' := A + D_A, B' := B + D_B, \tag{4}$$

where $D_A \in \Delta_A$ and $D_B \in \Delta_B$. Here, A and B are the nominal values of the matrices A' and B' of (3), respectively, while D_A and D_B represent perturbations and uncertainties. It is convenient to use the notation

$$D := (D_A, D_B) \text{ and } \Delta := \Delta_A \times \Delta_B, \tag{5}$$

so that $D \in \Delta$. We refer to Δ as the *uncertainty range*. The only information available about the system Σ are the nominal matrices A and B and the maximal uncertainty magnitude d; the entries of the matrices D_A and D_B are not known. Recalling the bound $M > 0$ of (2), our performance requirement becomes

$$e(t) := x^T(t)x(t) \le M \text{ for all } D \in \Delta \text{ and } t \in [0, t_f], \tag{6}$$

where x^T is the transpose of x. For the initial condition we require $x_0^T x_0 \le M$, so that performance is within bounds when the feedback channel is disconnected. Our objective is to find an input function $u(t)$ that maximizes the duration t_f.

Given two m-dimensional vector valued functions $a(t)$ and $b(t)$, define the weighted inner product

$$\langle a, b \rangle = \int_0^\infty e^{-\alpha t} a(t)^T b(t) dt \tag{7}$$

where $\alpha > 0$ is a real number; the integral is taken in the Lebesgue sense. The weight function $e^{-\alpha t}$ makes the inner product (7) well defined for all bounded functions. Denote by $L_2^{\alpha,m}$ the Hilbert space of all m-dimensional Lebesgue measurable functions with the inner product (7).

Physical systems are subject to input amplitude restrictions that are determined by the largest amplitude signal components can tolerate. To characterize these restrictions for a signal with m components, we use the point-wise ℓ^∞-norm

$$\|u(t)\| = max_{i=1,\ldots,m} |u_i(t)|.$$

Letting $K > 0$ be the input amplitude bound of the system Σ, all input functions of Σ must satisfy $\|u(t)\| \le K$ for all t. With the inner product (7), any bounded Lebesgue measurable function is a member of the Hilbert space $L_2^{\alpha,m}$. Denote by

$$U := \{u \in L_2^{\alpha,m} : \|u(t)\| \le K \text{ for all } t \ge 0\} \tag{8}$$

the set of all permissible input functions of Σ. Then, our objective is to find an input function $u \in U$ that drives the system Σ so as to satisfy the bound (6) for the longest possible time, irrespective of the perturbation of Σ.

A. Problem Statement

The state trajectory $x(t)$ of the system Σ depends, of course, on the perturbation matrices D_A and D_B, as well as on the input function u. Including these variables in explicit form in the function x, we often write $x(t, D, u)$ instead of $x(t)$, where $D = (D_A, D_B)$. Then, (6) takes the form

$$x^T(t, D, u)x(t, D, u) \le M \text{ for all } D \in \Delta \text{ and all } t \in [0, t_f]. \tag{9}$$

To characterize the time during which $x^T(t)x(t)$ does not exceed the bound M, we define the quantity

$$T(M, D, u) := \inf\{t \ge 0 : x^T(t, D, u)x(t, D, u) > M\}, \tag{10}$$

where $T(M, D, u) := \infty$ if $x^T(t)x(t) \le M$ for all $t \ge 0$. The fact that the initial state satisfies $x_0^T x_0 \le M$ implies that $T(M, D, u) \ge 0$. Referring to (6), we can see that $t_f = T(M, D, u)$ for the current selections of D and u. As we show later, the particular form of (10) guaranties that t_f is an upper semi-continuous functional in u, a mathematical fact that simplifies some of our forthcoming arguments.

Among the variables of the state trajectory $x(t, D, u)$, the entries of $D = (D_A, D_B)$ are unknown and unpredictable. As no feedback is available, the control input function u cannot depend on D. In order to guarantee that the bound (9) is valid for all possible D, we must consider the 'worst case' with respect to D. This leads us to the quantity

$$T^*(M, u) := \inf_{D \in \Delta} T(M, D, u), \tag{11}$$

which describes, for a particular u, the duration during which (9) is valid for all permissible perturbations D.

The duration $T^*(M,u)$ still depends on the input function u, and we can choose any input function in the set U of (8). Of course, the best choice is an input function u that maximizes $T^*(M,u)$; this yields the maximal duration

$$t_f^* := \sup_{u\in U} T^*(M,u). \tag{12}$$

Assuming that such an optimal input function exists, denote it by u^*, so that $t_f^* = T^*(M,u^*)$. The following statement, which is reproduced from [2], shows that such an optimal input function u^* does exist in the set U. The system Σ of (3) is *nominally unstable* if the nominal matrix A has at least one eigenvalue with positive real part.

Theorem 2. *Assume that the system Σ of (3) is nominally unstable and has a non-zero initial state, and let $T^*(M,u)$ be given by (11). Then, the following are valid.*

(i) There is a maximal time $t_f^* := \sup_{u\in U} T^*(M,u) < \infty$, and
(ii) There is a function $u^* \in U$ satisfying $t_f^* = T^*(M,u^*)$. □

Our objective is to characterize features of the optimal correction signal $u^*(t)$. As we can see from (11) and (12), the derivation of the optimal input function u^* involves the solution a max-min optimization problem. We explore some of the mathematical features of this problem in the next section.

III. Characteristics of Optimal Solutions

In the present section, we show that an optimal input function $u^*(t)$ is often a bang-bang function, i.e., a function whose components switch between the bounds $\pm K$. When $u^*(t)$ is not a bang-bang function, it is shown in [2] that it can be approximated by a bang-bang function. Being determined by their switching times, bang-bang functions are relatively easy to compute and implement.

A. Optimal solutions

To simplify somewhat the analysis of our Optimization Problem 1, it is convenient to reformulate the problem so as to turn the terminal time into a constant. This is achieved simply by introducing a time-scaling factor $\beta > 0$, so that the actual time t is expressed as

$$t = \beta s,$$

where the variable s has the fixed range $0 \le s \le 1$. Then, β represents the terminal time t_f of the process. To obtain the maximal time duration, we maximize the value of the scaling factor β . Denote

$$y(s) := x(\beta s) \text{ and } v(s) := u(\beta s), s \in [0,1],$$

and define the set of input functions

$$V := \left\{ v \in L_2^{\alpha,m} : \begin{array}{l} \|v(s)\| \le K \text{ for all } 0 \le s \le 1 \text{ and} \\ v(s) = 0 \text{ for all } s > 1 \end{array} \right\} \tag{13}$$

Denote $\dot{y} := dy(s)/ds$, so that $\dot{y} = \beta dx/dt$. Then, equation (3) takes the form

$$\Sigma : \dot{y}(s) = \beta[A'y(s) + B'v(s)], 0 \le s \le 1, y(0) = x_0, \tag{14}$$

where the matrices $A' = A + D_A$ and $B' = B + D_B$ are as in (4) and the input function $v(s)$ is taken from the set V of (13). Here, the new "time variable" s is within the fixed interval $[0,1]$, and is not subject to optimization.

As the solution $y(s)$ of (14) depends on the matrices $D := (D_A, D_B)$ as well as on the number β and on the input function v, we often denote it by $y(s;\beta,D,v)$. Recalling (6), we are then interested in values of β and in input functions $v \in V$ for which

$$y^T(s;\beta,D,v)y(s;\beta,D,v) \le M$$

for all $0 \le s \le 1$ and for all matrices $D \in \Delta$, given that the initial condition x_0 has a magnitude $x_0^T x_0 \le M$. A slight reflection shows that the maximal value of β is given by t_f^* of (12); denote $\beta^* := t_f^*$. When the system Σ is nominally unstable, it follows by Theorem 2 that the maximal value β^* exists and is bounded, and that there is an input function $v^*(s)$ that achieves this maximum. Using the notation of Theorem 2, we have

$$\begin{array}{c} v^*(s) := u^*(\beta^* s), 0 \le s \le 1, \\ \beta^* = t_f^*. \end{array} \tag{15}$$

Next, define the sets of matrices

$$\begin{array}{l} \{A + \Delta_A\} := \{A' \in R^{n\times n} : A' = A + D_A, D_A \in \Delta_A\}, \\ \{B + \Delta_B\} := \{B' \in R^{n\times m} : B' = B + D_B, D_B \in \Delta_B\}. \end{array}$$

To further shorten the notation, we use

$$\Xi := \{A + \Delta_A\} \times \{B + \Delta_B\}.$$

Let $\omega(s,A',B')$ be a Radon probability measure on the set

$$P := [0,1] \times \Xi. \tag{16}$$

Given a point $(s,A',B') \in P$, let $\omega(A',B'|s)$ be the corresponding conditional probability measure, and let $\omega(s)$ be the corresponding marginal probability measure, so that

$$\omega(s,A',B') = \omega(A',B'|s)\omega(s) \text{ for all } (s,A',B') \in P. \tag{17}$$

The next statement, whose proof is provided in section 4 below, is the main result of this note. It introduces a switching function for the optimal solution of Problem 1.

Theorem 3. *Under the conditions of Theorem 2, let $(v^*(s),\beta^*)$ be an optimal solution of Problem 1 as described by (15), and let V be the set of input functions (13). Then, there is a Lebesgue measurable function $z(s) : [0,1] \to R^m$ satisfying $z^T(s)v^*(s) \le z^T(s)v(s)$ for all input functions $v \in V$ and for almost all times $s \in [0,1]$.* □

Theorem 3 has the following important consequence: if a component of the function $z(s)$ is non-zero over an interval of time, then the corresponding component of the optimal input function $v^*(s)$ must equal either $+K$ or $-K$ over the same time interval, where K is the maximal input amplitude of the controlled system Σ. Indeed, assume that the j-th component $z_j(s)$ of $z(s)$ is positive over the interval $[s_1,s_2] \subset [0,1]$, and consider the measurable input function $v(s) \in V$ whose components are given by

$$v_i(s) := \begin{cases} -K \text{ if } i = j, \\ 0 \text{ if } i \ne j. \end{cases}$$

Then, the inequality $z^T(s)v^*(s) \leq z^T(s)v(s)$ of Theorem 3 becomes $z_j(s)v_j^*(s) \leq z_j(s)(-K)$; canceling $z_j(s) > 0$, we obtain $v_j^*(s) \leq -K$. As all input functions of Σ must have amplitude not exceeding K, the last inequality yields $v_j^*(s) = -K$ for all $s \in [s_1, s_2]$. When $z_j(s) < 0$ for all $s \in [s_1, s_2]$, a similar argument shows that $v_j^*(s) = K$ for all $s \in [s_1, s_2]$. We can summarize this discussion as follows.

Corollary 4. *Under the conditions of Theorem 3, assume that all components of the function $z(s)$ are non-zero almost everywhere in the interval $[0,1]$. Then, the optimal solution $v^*(s)$ of Problem 1 is a bang-bang function, where*

$$v_j^*(s) = \begin{cases} -K \text{ when } z_j(s) > 0, \\ K \text{ when } z_j(s) < 0, \end{cases}$$

for all $j = 1, 2, \ldots, m$ and almost all $s \in [0,1]$. □

Thus, the function $z(s)$ of Theorem 3 is reminiscent of the classical switching functions that appear in bang-bang control problems (e.g. [12]). We note, however, that no conclusion can be drawn about the optimal input function $v^*(s)$ in intervals over which $z(s)$ is identically zero. The form of $z(s)$ is:

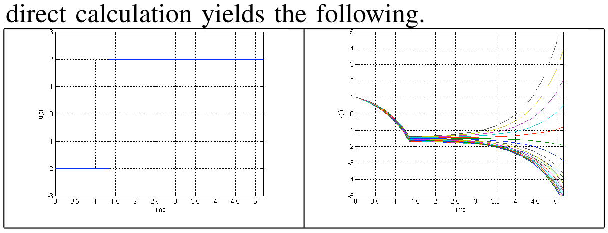
Corollary 5. *The function $z(s)$ of Theorem 3 can be expressed in the form*

$$z^T(s) = \int_s^1 \int_\Xi y(\zeta, A', B'; \beta^*, v^*))^T e^{\beta^* A'(\zeta - s)} B' \, d\omega(A', B'|\zeta) d\omega(\zeta); \quad (18)$$

here, $\omega(A', B', z)$ is a Radon probability measure with support $\Omega = \{(A', B', \zeta) \in \{A + \Delta_A\} \times \{B + \Delta_B\} \times [0,1]: y^T(\zeta, A', B'; \beta^*, v^*) y(\zeta, A', B'; \beta^*, v^*) = M\}$. □

The proof of Corollary 5 is provided in the next section.

Example 6. Consider the one-dimensional system $\dot{x}(t) = ax(t) + u(t)$, where $1.2 \leq a \leq 1.4$; $|u(t)| \leq 2$ for all t; $x(0) = 1$; and the output bound is $M = 25$. Using (18), it can be shown that $z(s) \neq 0$ for almost all $s \in [0,1]$ in this case (see [3, 4]). By Corollary 4, the optimal solution is then a bang-bang function. A direct calculation yields the following.



Optimal input	Response for various a

IV. Mathematical Deliberations

A. Preliminaries

The mathematical foundations of our discussion are rooted in the geometric form of the Hahn-Banach Theorem (e.g., [1], Ch. 2]), which can be stated as follows. Let S' and S'' be non-empty disjoint convex subsets of a topological vector space T, where the interior of S' is not empty. Then, there is a non-zero linear functional ℓ on T that separates S' and S'', namely, there is a real number ρ such that $\ell(s') \leq \rho \leq \ell(s'')$ for all $s' \in S'$ and all $s'' \in S''$.

We apply the Hahn-Banach Theorem on subsets of the cross product space $R \times B$, where R denotes the real numbers and B is a Banach space. For a subset $C \subset B$, denote by $\bar{C}$ the closure of C. Two projections are used below:

(i) The projection onto the reals $\Pi_r : R \times B \to R : (r, b) \mapsto r$, and

(ii) The projection $\Pi^- : R \times B \to B$ that singles out pairs with negative real parts:

$$\Pi^-(r, b) := \begin{cases} b \text{ if } r < 0, \\ \varnothing \text{ if } r \geq 0, \end{cases}$$

where $\varnothing$ is the empty set. The following auxiliary result is important for our ensuing discussion.

Lemma 7. *Let C be an open convex subset of the Banach space B, and let S be a convex subset of $R \times B$. Assume that S includes the zero 0, that 0 is an interior point of $\Pi_r S$, and that $0 \in \bar{C}$. Then, one of the following is true:*

(i) There is a non-zero linear functional $\ell : B \to R$ such that $\ell(c) \leq 0 \leq \ell(s)$ for all $c \in \bar{C}$ and all $s \in \Pi^- S$; or

(ii) There is an $s \in S$ for which $\Pi_r s < 0$ and $\Pi^- s \in C$.

Proof: (sketch) As (ii) is equivalent to the relation $C \cap \Pi^- S \neq \varnothing$, it only remains to show that (i) is valid when $C \cap \Pi^- S = \varnothing$. To this end, note that since S is convex, so is the projection $\Pi^- S$. Therefore, by the Hahn-Banach theorem, the equality $C \cap \Pi^- S = \varnothing$ implies that there is a non-zero linear functional $\ell : B \to R$ and a real number α such that $\ell(c) \leq \alpha \leq \ell(s)$ for all $s \in \Pi^- S$ and all $c \in C$. By the Lemma's assumptions, $0 \in S$ and $0 \in \bar{C}$; hence, $0 \in \bar{C} \cap \overline{\Pi^- S}$, and since ℓ is linear, we must have $\alpha = 0$. ■

Next, we review a generalized notion of the directional derivative, often referred to as the *Gateaux derivative*. Let X be a vector space over the real numbers R, let D be a subset of X, let V be a normed space, and let $T : D \to V$ be a function. Let $x, h \in D$ be two vectors, and assume that there is a real number $a(h) > 0$ such that $x + \alpha h \in D$ for all $0 < \alpha < a(h)$. Then, the *right-sided Gateaux derivative* $\mathfrak{D}T(x; h)$ of T at x in the direction h is the right derivative of $T(x + \alpha h)$ with respect to α, i.e.,

$$\mathfrak{D}T(x; h) := \lim_{\alpha \to 0^+} \frac{1}{\alpha} [T(x + \alpha h) - T(x)],$$

where the limit is taken in the norm of V. If the limit exists for all $h \in D$, then T is *right Gateaux differentiable* at x.

Further, the right Gateaux derivative of T at x is *linear* in its direction if

$$\mathfrak{D}T(x; \alpha h + \beta k) = \alpha \mathfrak{D}T(x; h) + \beta \mathfrak{D}T(x; k)$$

for all real numbers α and β and all valid directions h and k. Linearity is a rather common feature of the Gateaux derivative.

Next, for a function $T : S_1 \to S_2$, denote by T^{-1} the inverse set function of T, i.e., $T^{-1}[S] := \{s \in S_1 : Ts \in S\}$ for a set $S \subset S_2$. The following technical statement is critical to our discussion (compare to [13]).

Lemma 8. *Let Q be a convex subset of a Banach space, let F be a convex subset of the real numbers R, and let P be a compact subset of a finite dimensional metric space. Denote by $C(P,R)$ be the space of continuous functions $P \to R$. Let ε and M be two positive real numbers, and let $G(-\varepsilon,M)$ be the subset of $C(P,R)$ consisting of all functions whose image is in the interval $[-\varepsilon,M]$. Let $T_1 : Q\times F \to R$ and $T_2 : Q\times F \to C(P,R)$ be functions that satisfy the following conditions.*

(1) The restriction of T_1 to the set $(T_2)^{-1}G(-\varepsilon,M)$ attains a minimum at the point $(q^*,f^*) \in (T_2)^{-1}G(-\varepsilon,M)$.

(2) The functions T_1 and T_2 have right-sided Gateaux derivatives that are linear in their direction at the point (q^*,f^*).

(3) The image $T_2[Q\times F] \subset C(P,R)$ includes only bounded functions.

Then, there is a Radon probability measure ω over P and an ω-integrable function $\lambda : P \to R$ such that

(i) $|\lambda(p)| = 1$ almost everywhere with respect to the measure ω;

(ii) $\int_P \lambda(p)\mathfrak{D}T_2((q^*,f^*);(q,f)-(q^*,f^*))(p)d\omega(p) \geq 0$ for all $(q,f) \in Q\times F$

(iii) $\lambda(p)T_2(q^*,f^*)(p) = \max\{\lambda(p)(-\varepsilon),\lambda(p)M\}$ for ω-almost every $p \in P$. □

The proof of Lemma 8 depends on the following facts.

Lemma 9. *Under the conditions of Lemma 8, the subset of $R\times C(P,R)$ given by*

$$W(p) : = \bigcup_{(q,f)\in Q\times F} \big(\mathfrak{D}T_1((q^*,f^*);(q,f)-(q^*,f^*)), \mathfrak{D}T_2((q^*,f^*);(q,f)-(q^*,f^*))(p)\big) \quad (19)$$

is a convex set for every $p \in P$.

Proof: (sketch). The Lemma follows directly from the linearity assumption of Lemma 8(2) and from the fact that Q is convex. ■

Allowing the variable p in $W(p)$ of (19) to vary, we obtain the set of functions $P \to R\times C(P,R)$ given by

$$S : = W(\bullet) = \bigcup_{(q,f)\in Q\times F} \big(\mathfrak{D}T_1((q^*,f^*);(q,f)-(q^*,f^*)), \mathfrak{D}T_2((q^*,f^*);(q,f)-(q^*,f^*))(\bullet)\big). \quad (20)$$

For a subset B of a topological space, denote by $\mathrm{Int}(B)$ the interior of B, namely, the largest open set contained in B. The following is then true.

Lemma 10. *Under the conditions of Lemma 8, let S be the subset given by (20), and define the set of continuous functions*

$$C := \{f \in C(P,R) : f = g - T_2(q^*,f^*) \text{ and } g \in Int(G(-\varepsilon,M))\} \quad (21)$$

Then, following are valid:

(i) C is an open convex subset of $C(P,R)$;

(ii) S is a convex subset of $R\times C(P,R)$;

(iii) If there is a point $(q',f') \in Q\times F$ at which $\mathfrak{D}T_1((q^*,f^*);(q',f')-(q^*,f^*)) \neq 0$, then 0 is an interior point of $\Pi_r S$;

(iv) $0 \in \bar{C}$;

(v) If $h \in C$, then $\gamma h \in C$ for all $0 < \gamma < 1$.

Proof: (sketch). *(i)* The set C is a shift of $\mathrm{Int}(G(-\varepsilon,M))$, so we need only to show that $\mathrm{Int}(G(-\varepsilon,M))$ is convex. The latter follows from the fact that $G(-\varepsilon,M)$ is convex.

(ii) is a consequence of the linearity of the Gateaux derivatives (assumption 2 of Lemma 8) and of the fact that Q and F are convex sets.

(iii) follows directly from the linearity of the Gateaux derivatives (assumption 2 of Lemma 8).

(iv) Let $0 < \delta < 1$ be a real number. Then, the function $(1-\delta)T_2(q^*,f^*)$ is an interior point of $G(-\varepsilon,M)$. Whence, $\theta_\delta := (1-\delta)T_2(q^*,f^*) - T_2(q^*,f^*) = -\delta T_2(q^*,f^*)$ is in C. But then, since $0 = lim_{\delta\to 0}\theta_\delta$, we have that $0 \in \bar{C}$.

(v) Consider a function $h \in C$. By (21), we have that $h = z - T_2(q^*,f^*)$ for a function $z \in \mathrm{Int}(G(-\varepsilon,M))$. Now let $0 < \gamma < 1$, write $\gamma h = \gamma[z - T_2(q^*,f^*)]$, and denote $s := \gamma h + T_2(q^*,f^*) = \gamma z + (1-\gamma)T_2(q^*,f^*)$. Then, since $z \in \mathrm{Int}(G(-\varepsilon,M))$, we obtain for all $p \in P$ that $s(p) = \gamma z(p) + (1-\gamma)T_2(q^*,f^*)(p) < \gamma M + (1-\gamma)M = M$. Similarly, $s(p) = \gamma z(p) + (1-\gamma)T_2(q^*,f^*)(p) > \gamma(-\varepsilon) + (1-\gamma)(-\varepsilon) = -\varepsilon$. Thus, $s \in \mathrm{Int}(G(-\varepsilon,M))$, and, since $\gamma h = s - T_2(q^*,f^*)$, it follows that $\gamma h \in C$. ■

Lemma 11. *Under the conditions of Lemma 8, assume that $\mathfrak{D}T_1((q^*,f^*);(q,f)-(q^*,f^*))$ is not the zero function, and let C be given by (21). Then, there is a linear functional $\ell : C(P,R) \to R$, not identically zero, such that*

$$\ell(\mathfrak{D}T_2((q^*,f^*);(q,f)-(q^*,f^*))(\bullet)) \geq 0 \text{ for all } (q,f) \in Q\times F \quad (22)$$

and

$$\ell(c) \leq 0 \text{ for all } c \in \bar{C}. \quad (23)$$

Proof: (sketch) In view of Lemma 10, the conditions of Lemma 7 are satisfied and whence one of the alternatives listed in Lemma 7 must be valid. Alternative (i) of Lemma 7 yields (22) and (23). By using Lemmas 10 and 8, it can be shown that alternative (ii) of Lemma 7 is invalid here, since it contradicts assumption (1) of Lemma 8 (see [3], [4] for details). ■

Proof: (of Lemma 8 (sketch)). Applying the Riesz - Markov Representation Theorem to the non-zero functional $\ell : C(P,R) \to R$ of Lemma 11 and using (22) and (23), we conclude that there is a positive Radon probability measure ω on P and an ω-integrable function $\lambda : P \to R$ such that

$$\begin{cases} |\lambda(p)| = 1 \text{ for } \omega\text{-almost all } p \in P; \\ \ell(c) = \int_P \lambda c d\omega \text{ for all functions } c \in C(P,R); \text{ and} \\ \omega(P) > 0. \end{cases} \quad (24)$$

Combining (24) with (22) yields Lemma 8*(ii)*.

Next, denote $c^*(\bullet) := T_2(q^*,f^*)(\bullet) : P \to R$. Then, in view of (21), every element $c \in \bar{C}$ can be written in the form $c = g - c^*$, where $g \in G(-\varepsilon,M)$. Consequently, (23) takes the form $\ell(g - c^*) \leq 0$ for $g \in G(-\varepsilon,M)$. By (24), the latter can be rewritten in the form

$$\int_P \lambda(p)[c^*(p) - c(p)]d\omega(p) \geq 0.$$

Finally, using the fact that ω is a Radon probability measure, it can be shown that the last inequality implies Lemma 8*(iii)* (see [3], [4] for details). ■

Proof: (of Theorem 3 (sketch)). The proof is based on Lemma 8. First, introduce the notation

$$Q := \{v(s) \in V : v(s) = 0 \text{ for } s > 1\},$$
$$F := [0, t_f^* + 1] \subset R,$$

where t_f^* is the maximal time of (12). Let P be the set given by (16). It can then be seen that Q is a convex subset of the Banach space $L_2^{\alpha,m}$; that F is a convex set of real numbers; and that P is a compact subset of the metric space $R^{(1+nn+mn)}$. Thus, Q, F, and P fulfill the requirements of the corresponding quantities listed in Lemma 8.

Recalling the set of matrices D of (5), define the functions

$$T_1(v(s),\beta) := -\beta, \text{ and}$$
$$T_2(v(s),\beta,p) := y^T(s;\beta,D,v(s))y(s;\beta,D,v(s)), \text{ where } p \in P.$$

A direct examination shows that the functions T_1 and T_2 satisfy the requirements of Lemma 8. Using the value β^* of (15), a direct calculation yield the Gateaux derivatives

$$\mathfrak{D}T_1((v^*,\beta^*);(v,\beta)-(v^*,\beta^*)) = \beta^* - \beta.$$

$$\mathfrak{D}T_2((v^*,\beta^*),(s,A',B');(v-v^*))|_{\beta=\beta^*} =$$
$$2y^T(s,A',B';\beta^*,v^*)\int_0^s e^{\beta^*A'(s-\tau)}\beta^*B'(v(\tau)-v^*(\tau))d\mu(\tau). \quad (25)$$

Denote $p := (\tau, A', B')$. Then, by Lemma 8, there is a Radon probability measure ω over P an ω-integrable function λ : P → R such that

$$|\lambda(p)| = 1 \text{ for } \omega\text{-almost all points } p \in P, \text{ and}$$
$$\int_P \lambda(p)\mathfrak{D}T_2\big((v^*,\beta^*),p;(v,\beta)-(v^*,\beta^*)\big)d\omega(p) \geq 0.$$

Next, denote $y^*(s) := y(s,A',B';\beta^*,v^*)$; set $\beta = \beta^*$; substitute (30) into the last inequality; define the function

$$\eta(s,\tau) := \begin{cases} 1 \text{ for } 0 \leq \tau \leq s, \\ 0 \text{ otherwise,} \end{cases} \quad (26)$$

where $s \geq 0$; and apply Fubini's Theorem. This yields

$$\int_0^1 \left\{ \int_P \lambda(p)(y^*(s))^T e^{\beta^*A'(s-\tau)}B'\eta(s,\tau)d\omega(p) \right\}(v(\tau)-v^*(\tau))\, d\mu(\tau) \geq 0. \quad (27)$$

Now, define the function

$$z^T(\tau) := \int_P \lambda(p)(y^*(s))^T e^{\beta^*A'(s\ -\tau)}B'\eta(s,\tau)d\omega(p). \quad (28)$$

Then, inequality (27) can be rewritten as

$$\int_0^1 z^T(\tau)(v(\tau)-v^*(\tau))d\mu(\tau) \geq 0 \text{ for all } v \in V. \quad (29)$$

Using the conditional and marginal measures of (17) together with (26), equation (28) can be rewritten as

$$z^T(\tau) = \int_\tau^1 \int_\Xi \lambda(s,A',B')(y^*(s))^T e^{\beta^*A'(s-\tau)}B'\, d\omega(A',B'|s)d\omega(s). \quad (30)$$

Finally, (29) implies the inequality

$$z^T(\tau)v(\tau) \geq z^T(\tau)v^*(\tau), \quad (31)$$

since otherwise one can build a function $v'' \in V$ that violates (29) when selecting $v := v''$ (see [3], [4] for details). Theorem 3 follows then by switching the names of τ and s. ■

Proof: (of Corollary 5 (sketch)). By Lemma 8*(iii)*, we have $\lambda(p)y^T(p;v^*,\beta^*)y(p;v^*,\beta^*) = \max_{a\in[-\varepsilon,M]}\lambda(p)a$. Now, for $\lambda(p) = 1$, the right side is M, so $y^T(p;\beta^*,v^*)y(p;\beta^*,v^*) = M$; for $\lambda(p) = -1$, the right side is $\varepsilon > 0$ while the left side is negative; hence, $\lambda(p) = -1$ is incompatible. Thus, $\lambda(p) = 1$ for ω-almost every $p \in P$, so that ω has the support $\Omega = \{p \in P : y^T(p;\beta^*,v^*)y(p;\beta^*,v^*) = M\}$. Finally, (18) follows from (30) by substituting $\lambda(p) = 1$ and switching the names of τ and s. ■

REFERENCES

[1] N. Bourbaki, "Topological Vector Spaces", Springer Verlag, Berlin, 1987.

[2] D. Chakraborty and J. Hammer, "Preserving system performance during feedback failure", Proceedings of the IFAC World Congress, Seoul, Korea, July 2008.

[3] D. Chakraborty, "Need Based Feedback: An Optimization Approach", Ph.D. Dissertation, University of Florida, 2007.

[4] D. Chakraborty and J. Hammer, "Optimal Control During Feedback Failure", International J. of Control (to appear), 2009.

[5] R.V. Gamkrelidze, "On some extremal problems in the theory of differential equations with applications to the theory of optimal control", J. SIAM, Ser. A Control 3, pp. 106-128, 1965.

[6] D.L. Kelendzheridze, "On the theory of optimal pursuit", Soviet Math. Dokl., 2, 1961, pp. 654-656, 1961.

[7] L.A. Montestruque and P.J. Antsaklis, "Stability of model-based networked control systems with time-varying transmission times," IEEE Transactions on Automatic Control, Vol. 49, No. 9, pp.1562-1572, 2004.

[8] G.N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: an overview", Proceedings of the IEEE, Vol 95(1), pp. 108-137, Jan 2007

[9] L.W. Neustadt, "An abstract variational theory with applications to a broad class of optimization problems I, general theory", SIAM. J. Control, 4, pp. 505-527, 1966.

[10] L.W. Neustadt, "An abstract variational theory with applications to a broad class of optimization problems II, applications", SIAM. J. Control, 5, pp. 90-137, 1967.

[11] J.C. Panetta and K.R. Fister, "Optimal control applied to competing chemotherapeutic cell-kill strategies", SIAM J. on Applied Mathematics, Vol. 63, No. 6, 2003

[12] L.S. Pontryagin, V.G. Boltyansky, R.V. Gamkrelidze, and E.F. Mishchenko, "The Mathematical Theory of Optimal Processes", Interscience Publishers John Wiley & Sons Inc., New York-London, 1962

[13] J. Warga, "Optimal Control of Differential and Functional Equations", Academic Press, New York, 1972.

[14] P.V. Zhivogyladov and R.H. Middleton, "Networked control design for linear systems", Automatica, 39, pp. 743-750, 2003.