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Bang-Bang Functions: Universal Approximants for the Solution of Min-Max Optimal Control Problems

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Abstract

Quite often, solutions of optimal control problems yield intricate input functions that are difficult to calculate and implement. Bang-bang functions are functions whose components assume only extremal values, switching from one extremal value to the other as necessary. Being entirely determined by their switching times, bang-bang functions are relatively easy to calculate and implement. In the present note, we consider the use of bang-bang functions to approximate solutions of a rather general class of optimization problems. Approximation here is in the sense of finding a simple input function that yields performance close to optimal performance, and not necessarily in the sense of finding a simple input function that approximates the optimal input function.

Our focus is on a rather general min-max optimization problem for systems that are subject to parameter uncertainties and disturbance signals. Specifically, consider a system Σ described by a differential equation of the form

$$\dot{x}(t) = f(x(t), v(t), u(t)), \tag{1}$$

where f is a continuous function, $x(t) \in R^n$ is the state of the system, $v(t) \in R^p$ is a disturbance signal, and $u(t) \in R^m$ is the control input signal of Σ. To avoid issues related to stability, we restrict our attention to optimization over a finite time interval $[0, t_f]$.

As is the case for most practical systems, the input signals of Σ are of bounded amplitude, with the amplitude bound being determined by the physical characteristics of Σ. In formal terms, there is a box $S \subset R^m$ such that all permissible input functions of Σ must satisfy $u(t) \in S$ for all $t \in [0, t_f]$. In addition, we require all input functions u of Σ to be Lebesgue measurable. Denoting by U the set of all Lebesgue measurable functions with values in S, we have that U describes the set of all permissible input functions of Σ. Similarly, denote by V the set of all permissible disturbance signals v of Σ. Finally, a constraint is imposed on the output of Σ in the form of a set Q of desirable output functions; only input functions $u \in U$ that satisfy the requirement $\Sigma u \in Q$ are desired.

In practice, the function f that describes the differential equation of Σ usually depends on parameters whose values are not accurately known. Let Σ_0 denote the version of the system Σ obtained when these parameters are all at their nominal values and the disturbance signal v is zero. Let $\Sigma_{\epsilon,v}$ be the system that results when the parameters of Σ experience a perturbation ϵ from their nominal values, while an external disturbance signal $v(t)$ is present. The exact values of the perturbation ϵ and of the disturbance signal v are not known. The only information provided is that $\epsilon \in E$ and $v \in V$, where E and V are specified compact sets.

Consider now an optimization problem for the system Σ with a cost function J: the objective is to find an input function $u \in U$ that minimizes the value $J(\Sigma u)$. To account for the perturbations and the disturbances, the optimal function u is selected so that it minimizes J for the 'worst' instances of perturbation and disturbance. The optimal value J^o of J is then

$$J^o = \inf_{u \in U} \sup_{\epsilon \in E, v \in V} \{J(\Sigma_{\epsilon,v} u) : \Sigma_{\epsilon,v} u \in Q\}. \tag{2}$$

Assume that there is an optimal input function $u^* \in U$ that satisfies (2). Given a number $\delta > 0$, denote by $N_\delta(Q)$ a neighborhood of radius δ around the constraint Q. Then, $N_\delta(Q)$ is the set of all state trajectories $x(t), t \in [0, t_f]$, that deviate by no more than δ from the constraint Q. In these terms, we show that the performance achieved by the optimal input function u^* can be approximated by a bang-bang input function, as follows.

THEOREM. For every real number $\delta > 0$, there is a bang-bang input function $u^{\pm}(t) \in U$ such that $|J(\Sigma_{\epsilon,v} u^*) - J(\Sigma_{\epsilon,v} u^{\pm})| \leq \delta$ and $\Sigma_{\epsilon,v} u^{\pm} \in N_\delta(Q)$ for all $\epsilon \in E$ and all $v \in V$. $\diamondsuit$

As we can see, the bang-bang input function $u^{\pm}$ approximates the performance obtained from the optimal input function u^* to any desirable accuracy $\delta > 0$. Reduction of the error δ comes at the cost of increasing the number of switches of the bang-bang function $u^{\pm}$.

Bang-bang input functions offer significant advantages in computation and implementation, as bang-bang functions are determined solely by their switching times. The use of bang-bang approximations reduces the complex problem of computing and implementing an optimal input function into a search and implementation of the switching times of $u^{\pm}$. An application of this result to a specific min-max optimization problem can be found in [1].

References

[1] D. Chakraborty and J. Hammer. Preserving system performance during feedback failure. *Proceedings of the IFAC World Congress*, Seoul, Korea, July 2008 (to appear).