INT. J. CONTROL, 1988, VOL. 48, NO. 3, 1183–1212

Assignment of dynamics for non-linear recursive feedback systems

JACOB HAMMER†

The problem of stabilizing a non-linear recursive system Σ using a closed-loop configuration with compensation and feedback is considered. Most attention is devoted to the design of the dynamical behaviour of the stabilized closed-loop configuration. It is shown that, except for some obvious restrictions, any stable dynamics can be assigned to the closed-loop system. Compensators that yield the desired dynamics are explicitly constructed in implementable recursive form, and their formulae are expressed in terms of quantities derived directly from the given descriptions of the system Σ and the desired dynamics. It is assumed that the state of the system Σ is accessible. The resulting closed-loop configurations are internally stable.

1. Introduction

Perhaps one of the most fundamental questions in control theory is: 'How, and to what extent, can the dynamical behaviour of a given system be altered from an undesirable, possibly unstable, behaviour into a desirable one through closed-loop feedback control?' For the case of linear control systems, this question has been answered in detail by various versions of the pole assignment theorems (see Wonham 1967, Rosenbrock 1970, Hammer 1983 b and other references cited in these works). However, it seems that for the case of non-linear systems, the question has remained largely unanswered. In this paper, the problem of altering the dynamical behaviour of a non-linear recursive system using a closed-loop configuration containing compensators and feedback is considered. By properly designing the compensators in the loop, it is shown that any dynamical behaviour can be achieved for the closed-loop system, subject only to some obvious limitations described in detail in § 3. Moreover, the compensators derived consist of recursive systems, and the resulting closed loop is internally stable, so the configuration can be implemented in practice using digital computers. Our discussion is limited to the case where the system Σ, whose dynamical behaviour is to be altered, has its state as output. Explicit formulae for compensators that yield the desired dynamics of the closed-loop system are derived, and these formulae are stated directly in terms of quantities that appear in the standard descriptions of the system Σ and the desired dynamics.

In qualitative terms, a discrete-time system is said to be 'recursive' whenever its output sequence can be computed from its input sequence in a recursive manner. To be more precise, let Σ be a discrete-time system, let $\{u_0, u_1, \ldots\}$ be an input sequence of Σ, and let $\{y_0, y_1, \ldots\}$ be the generated output sequence. The system Σ is said to be recursive if there is a pair of integers, $\eta, \mu \geqslant 0$, and a function f, such that any output sequence $\{y_0, y_1, \ldots\}$ can be computed from the input sequence $\{u_0, u_1, \ldots\}$ generating

Received 11 November 1987.

† Center for Mathematical System Theory, Department of Electrical Engineering, University of Florida, Gainesville, FL 32611, U.S.A.

it by using a recursive relation of the form

$$y_{k+\eta+1}=f(y_k,\ldots,y_{k+\eta},u_k,\ldots,u_{k+\mu}),\quad k=0,1,2,\ldots$$

The initial conditions, $y_0, \ldots, y_\eta$, of the system have to be pre-specified. The function f is referred to as a 'recursion function' for the system Σ. The advantage of working with recursive systems is, of course, that they can be implemented directly on digital computers, both singly and in combination.

In this paper general recursive systems are not discussed. Instead, attention is limited to a particular class of recursive system having a recursive representation of the form

$$x_{k+1}=f(x_k,u_k),\quad k=0,1,2,\ldots \tag{1.1}$$

with the initial condition x_0 given. A representation of the form (1.1) is usually called a 'state representation', and it is assumed that the systems Σ, whose dynamical behaviour we wish to alter, are given in terms of their state representations; such systems are quite common. Indeed, one might say that most systems encountered in engineering practice are either given in terms of a recursive representation of this form, or such a representation can be obtained for them by providing access to some additional variables of the system. Of course, one could also employ observer techniques to gain access to the state x of a given system when the state is not provided as output, but this point is not elaborated upon in this paper. The solution to the dynamics assignment problem takes a somewhat simpler form when the system that needs to be altered is given in terms of a representation of the form (1.1), and so this representation is adopted throughout this paper. The more general case of recursive systems is discussed in a separate report.

The basic control configuration used for altering the dynamics of a given system Σ is a closed feedback loop containing two compensators, a precompensator and a feedback compensator. The objective is to construct precompensators and feedback compensators that, when connected in a closed loop around the system Σ, yield an (internally) stable configuration which has the specified input/output dynamics. Such compensators are referred to as 'stabilizing compensators' for the system Σ, even though stabilization is not the only consideration. The stabilizing compensators constructed are combinations of recursive systems, and explicit formulae are derived for them in terms of the given recursion function f of the system Σ and the given recursive description of the dynamics desired for the closed-loop system. Being combinations of recursive systems, these compensators can then be implemented on digital computers. This paper is a continuation of the work on stabilization of non-linear systems reported by Hammer (1984 a, b, 1985 a, b, 1986, 1987), and the discussion begins with a few words regarding some basic notions of the theoretical set-up.

Of crucial importance to the set-up is the theory of fraction representations of non-linear systems developed by Hammer (1985 a, 1986, and in particular 1987). Briefly, given a non-linear system Σ, two types of fraction representations are constructed: a right fraction representation, which is of the form $\Sigma = PQ^{-1}$, where P and Q are stable systems; and a left fraction representation, which is of the form $\Sigma = G^{-1}T$, where G and T are stable systems. As will be seen throughout the discussion, right and left fraction representations are extremely useful in the development of a stabilization theory for non-linear systems, in close analogy to their usefulness in the development of the theory of stabilization for linear systems.

As noted in the present author's previous reports, it is particularly convenient to stabilize non-linear systems using the control configuration shown in Fig. 1, where Σ is the given system, π is a precompensator, and φ is a feedback compensator. The overall closed-loop system is denoted by $\Sigma_{(\pi,\varphi)}$. It is particularly convenient to choose the compensators π and φ in such a way that π is the inverse of a stable system and φ is a stable system, so that

$$\left.\begin{aligned} \pi &= B^{-1} \\ \varphi &= A \end{aligned}\right\} \tag{1.2}$$

where A and B are stable systems, B is invertible, and A and B^{-1} are causal systems.

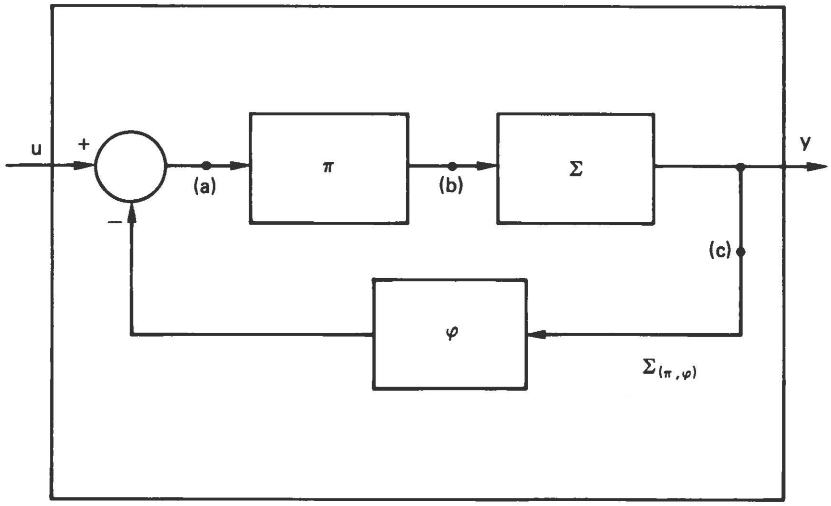

Figure 1.

Now assume that the compensators π and φ were chosen in accordance with (1.2), and that the given system Σ has a right fraction representation $\Sigma = PQ^{-1}$. Then it is easy to see that (under some mild conditions, e.g. when Σ is strictly causal) the input/output relation induced by the composite system $\Sigma_{(\pi,\varphi)}$ is given by

$$\begin{aligned} \Sigma_{(\pi,\varphi)} &= \Sigma\pi[I + \varphi\Sigma\pi]^{-1} = P(BQ)^{-1}[I + AP(BQ)^{-1}]^{-1} \\ &= P[AP + BQ]^{-1} \end{aligned} \tag{1.3}$$

Denoting

$$M := [AP + BQ] \tag{1.4}$$

it follows that

$$\Sigma_{(\pi,\varphi)} = PM^{-1} \tag{1.5}$$

so that the composite system $\Sigma_{(\pi,\varphi)}$ is input/output stable whenever the stable system M has a stable inverse M^{-1}. As will be mentioned later, in somewhat more stringent circumstances, this would also guarantee that the composite system $\Sigma_{(\pi,\varphi)}$ is internally stable, where internal stability in this case means that the stability of the closed-loop configuration is not destroyed by small additive noises that may appear at the ports of the systems of which it consists. Thus, the problem of stabilizing the system Σ reduces to the problem of finding a pair of stable systems A and B for which the combination $AP + BQ$ has a stable inverse—and where the systems A and B satisfy the additional conditions alluded to earlier. A stable system M that also has a stable inverse M^{-1} is called a 'unimodular' system.

Another important qualitative consequence of (1.5) is that the input/output dynamical behaviour of the closed-loop system $\Sigma_{(\pi,\varphi)}$ can be controlled through the unimodular system M. Thus, the possibilities of assigning the dynamics of the closed-loop system hinge on the extent to which we have freedom in the selection of M. The main objectives in this paper are to find the restrictions on the unimodular systems M that can be selected; to find appropriate systems A and B satisfying $AP + BQ = M$; and to find explicit representations of the systems A and B in recursive form, expressed in terms of the given recursion function f of the system Σ and the given description of M. In § 3 it will be shown that there are very few restrictions on M. In somewhat simplistic form, it can be said that M may be chosen as any unimodular and bicausal recursive system. For each permissible choice of M, systems A and B satisfying $AP + BQ = M$, are derived in § 3. These systems consist of combinations of recursive systems, and they satisfy all the conditions for an internally stable implementation. Thus, it can be stated qualitatively that arbitrary dynamics can be assigned.

The discussion in this paper heavily depends on the theory of stabilization for non-linear systems developed by Hammer (1986, 1987). It will be assumed, as it was then, that the final closed-loop system $\Sigma_{(\pi,\varphi)}$ is operated only by bounded input sequences, namely, by input sequences whose amplitudes do not exceed a prespecified value θ. The particular value of θ is, however, immaterial, and it can be chosen arbitrarily.

The paper is organized as follows. In § 2 the general theoretical framework of Hammer (1986, 1987), which forms the basis of our developments in the present paper, is reviewed, refined and extended. The main results of the paper are contained in § 3, where an explicit solution is provided to the problem of dynamics assignment for a non-linear system having a recursive representation of the form (1.1).

This introduction is concluded with a few words regarding the background literature concerning the stabilization of non-linear systems. Notably, it is beyond the scope of this work to provide an extensive overview of the literature. This paper is based on the work of Hammer (1984 a, b, 1985 a, b, 1986, 1987). The present approach to the theory of non-linear systems has been strongly influenced by the transfer matrix theory of linear systems as presented by Rosenbrock (1970), Desoer and Chan (1975), Hammer (1983 a, b), the references cited in these works, and other related expositions. Alternative recent treatments involving the stabilization of non-linear systems have been presented by Vidyasagar (1980), Sontag (1981), Desoer and Lin (1984), the references listed in these papers, and many others.

2. Preliminaries

The systems considered in this paper are discrete-time non-linear systems, accepting input sequences of m-dimensional real vectors, and generating output sequences of p-dimensional real vectors, where m and p are arbitrary positive integers. In order to describe these systems in more accurate terms, let $m > 0$ be an integer. Denote the set of real numbers by $\mathbb{R}$, the set of all m-dimensional real vectors by $\mathbb{R}^m$, and the set consisting of only the zero element 0 by $\mathbb{R}^0$. The set of all sequences $u_0, u_1, u_2, \ldots$, where $u_i \in \mathbb{R}^m$ for all integers $i \geqslant 0$, is denoted by $S(\mathbb{R}^m)$. Given a sequence $u \in S(\mathbb{R}^m)$ and an integer $i \geqslant 0$, the ith element of the sequence is denoted by u_i, and the index i is interpreted as the time marker. For a pair of integers $j \geqslant i \geqslant 0$ and a sequence $u \in S(\mathbb{R}^m)$, the elements $u_i, u_{i+1}, \ldots, u_j$ are denoted by u_i^j.

A system Σ is simply a map $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$, transforming m-dimensional input sequences into p-dimensional output sequences. It will be assumed that the system Σ is completely described by its input/output relationship (i.e. by the map it induces), so that it does not possess any internal instabilities not reflected in the input/output relation. This is a standard assumption, without which even linear systems cannot be stabilized. Given a system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$, a sequence $u \in S(\mathbb{R}^m)$, and a pair of integers $j \geqslant i \geqslant 0$, the ith element of the output sequence $y := \Sigma u$ is denoted by $\Sigma u]_i$, and the elements $y_i, y_{i+1}, \ldots, y_j$ of that output sequence are denoted by $\Sigma u]_i^j$.

For the sake of completeness, the operation of addition for sequences and for systems is now reviewed briefly. For sequences, let $u = \{u_0, u_1, u_2, \ldots\}$ and $v = \{v_0, v_1, v_2, \ldots\}$ be a pair of sequences in $S(\mathbb{R}^m)$. The sum $w := u + v$ is defined elementwise, so that $w_i = u_i + v_i$ for all integers $i \geqslant 0$. Given two systems $\Sigma_1, \Sigma_2 : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$, the system $\Sigma := \Sigma_1 + \Sigma_2 : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ consisting of their sum is defined pointwise, so that for every input sequence $u \in S(\mathbb{R}^m)$, the output sequence $y := \Sigma u$ is given by $y = \Sigma_1 u + \Sigma_2 u$. These are, of course, the standard operations of addition used in this context in the literature.

Given a vector $v := (v^1, \ldots, v^m) \in \mathbb{R}^m$, the maximal absolute value of its coordinates is denoted by $|v| := \max\{|v^1|, \ldots, |v^m|\}$. For a real number $\theta > 0$, let $S(\theta^m)$ be the set of all sequences $u = \{u_0, u_1, u_2, \ldots\} \in S(\mathbb{R}^m)$ satisfying $|u_i| \leqslant \theta$, for all integers $i \geqslant 0$. Thus, $S(\theta^m)$ can be interpreted as the set of all input sequences bounded by θ. A sequence $u \in S(\mathbb{R}^m)$ is said to be 'bounded by' θ if $u \in S(\theta^m)$, and u is said to be 'bounded' if there is a real $\theta > 0$ such that $u \in S(\theta^m)$. Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a system. For a subset $S \subset S(\mathbb{R}^m)$, denote by $\Sigma[S]$ the image of the set S through Σ, namely, the set of all output sequences generated by input sequences belonging to the set S. When S is the entire domain $S(\mathbb{R}^m)$ of the system Σ then $\Sigma[S] =: \operatorname{Im} \Sigma$ denotes the image of Σ. Also, given a subspace $S' \subset S(\mathbb{R}^p)$, denote by $\Sigma^*[S']$ the inverse image of the set S' through Σ, namely, the set of all input sequences in $S(\mathbb{R}^m)$ which are mapped by Σ into the set S'. As usual, a system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is said to be BIBO- (Bounded-Input Bounded-Output) stable if, for every real $\theta > 0$, there is a real $N > 0$ such that $\Sigma[S(\theta^m)] \subset S(N^p)$. The term 'bounded system' will be used for a system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ for which there is a real number $N > 0$ such that $\operatorname{Im} \Sigma \subset S(N^p)$, namely, for a system having all its output sequences bounded by N. Given a sequence $u \in S(\mathbb{R}^m)$, denote $|u| := \sup\{|u_i|, i = 0, 1, 2, \ldots\}$, which may possibly be infinite and $\rho(u) := \sup\{2^{-i}|u_i|, i = 0, 1, 2, \ldots\}$, which may also become infinite. The norm $|\cdot|$ is the usual l^∞ norm, and the norm $\rho(\cdot)$ is sometimes called a 'weighted l^∞ norm'. Using the norm ρ, a metric ρ is induced on the space $S(\mathbb{R}^m)$ by defining $\rho(u, v) := \rho(u - v)$. Whenever discussing continuity over the space $S(\mathbb{R}^m)$ or any of its subsets, continuity with respect to the topology induced by the metric ρ is inferred unless explicitly stated otherwise.

The definition of the notion of stability that will be employed throughout the discussion can now be reviewed. Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a system. Σ is said to be a 'stable' system if it is BIBO-stable, and if, for every real $\theta > 0$, the restriction $\Sigma : S(\theta^m) \to S(\mathbb{R}^p)$ is a continuous map. This definition of stability is in the spirit of the classical definition of stability due to Lyapunov. To deal with stable systems possessing stable inverses, the following terminology will be used. Let $S_1 \subset S(\mathbb{R}^m)$ and $S_2 \subset S(\mathbb{R}^p)$ be a pair of subspaces, and let $M : S_1 \to S_2$ be a system. The system M is said to be a 'unimodular' system if it is invertible, and if M and M^{-1} are both stable systems. In case there is a unimodular system $M:S_1 \to S_2$, the spaces S_1 and S_2 are said to be 'stability-morphic'.

The notion of causality is of major importance to control theory since, as is well known, only causal systems can be implemented in real-time applications. Some standard terminology in this context is now reviewed. A system $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is 'causal' (respectively, 'strictly causal') if, for every pair of input sequences $u, v \in S(\mathbb{R}^m)$, and for every integer $i \geq 0$, the equality $u_0^i = v_0^i$ implies that $\Sigma u]_0^i = \Sigma v]_0^i$ (respectively, $\Sigma u]_0^{i+1} = \Sigma v]_0^{i+1}$). A system $M: S(\mathbb{R}^m) - S(\mathbb{R}^m)$ is 'bicausal' if it is invertible and if M and M^{-1} are both causal systems.

It is assumed that the systems Σ whose stabilization is considered are all strictly causal. When the system Σ is recursive, strict causality is simply equivalent to the requirement that Σ possesses a recursive representation $y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu})$ in which $\eta \geq \mu$. In other words, for a strictly causal recursive system, there is no direct coupling from the input to the output. This assumption is not really critical to the theory here, but its use shortens and simplifies some of the arguments. Every recursive system with a representation of the form $x_{k+1} = f(x_k, u_k)$ is, of course, strictly causal. When discussing the stabilization of a strictly causal system Σ it will be more convenient to consider, qualitatively speaking, the stabilization of the system $\gamma + \Sigma$, where γ is a constant non-singular matrix, instead of considering the stabilization of the system Σ directly. The reason for this is that the system $\gamma + \Sigma$ is injective (one to one) whenever Σ is strictly causal, and it is very easy to (left-) invert in the recursive case (see (2.2) and Proposition 1). The significance of the constant non-singular matrix γ will be discussed in § 3. When the system $\gamma + \Sigma$ is stabilized using the control configuration in Fig. 1, stabilization of the original system Σ is also obtained (see the Corollary at the end of § 2) in a control configuration which is slightly different from that in Fig. 1, and which is given in Fig. 2.

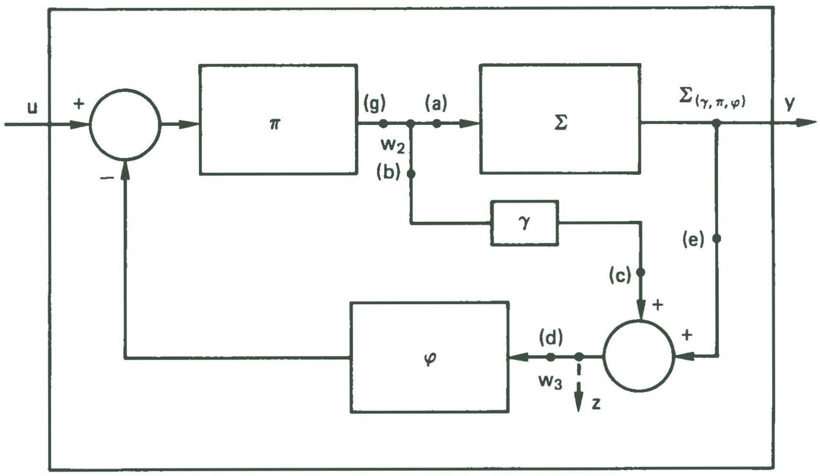

Figure 2.

In Fig. 2, the input is u and the output is y, the block γ represents a static transformation invoked by the matrix γ, and the overall system described by this configuration is denoted by $\Sigma_{(\gamma,\pi,\varphi)}$. In order to allow for possible discrepancy in the dimensions of the input space and the output space of the system Σ, we formally construct the quantity '$\gamma + \Sigma$' as follows.

Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ be a strictly causal system. Let $p := \max\{m, q\}$, and define the identity injection maps $\mathscr{I}_1 : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ and $\mathscr{I}_2 : S(\mathbb{R}^q) \to S(\mathbb{R}^p)$ as follows. If $q \geqslant m$, write

$$S(\mathbb{R}^p) = S(\mathbb{R}^q) = S(\mathbb{R}^m) \times S(\mathbb{R}^{q-m})$$

let

$$\mathscr{I}_1 : S(\mathbb{R}^m) \to S(\mathbb{R}^p) : \mathscr{I}_1[S(\mathbb{R}^m)] = S(\mathbb{R}^m) \times 0$$

be the obvious identity injection, and let

$$\mathscr{I}_2 : S(\mathbb{R}^q) \to S(\mathbb{R}^p) \quad (= S(\mathbb{R}^q))$$

be the identity map. If $q < m$, write

$$S(\mathbb{R}^p) = S(\mathbb{R}^m) = S(\mathbb{R}^q) \times S(\mathbb{R}^{m-q})$$

let

$$\mathscr{I}_2 : S(\mathbb{R}^q) \to S(\mathbb{R}^p) : \mathscr{I}_2[S(\mathbb{R}^q)] = S(\mathbb{R}^q) \times 0$$

be the obvious identity injection, and let

$$\mathscr{I}_1 : S(\mathbb{R}^m) \to S(\mathbb{R}^p) \quad (= S(\mathbb{R}^m))$$

be the identity map. Then, the system

$$\Sigma_\gamma := \gamma \mathscr{I}_1 + \mathscr{I}_2 \Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p) \tag{2.1}$$

where γ is a $p \times p$ constant non-singular matrix, is injective by the strict causality of the system Σ (see Proposition 1 and the discussion preceding it). The injections $\mathscr{I}_1$ and $\mathscr{I}_2$ are rather easy to implement. For instance, when $q \geqslant m$, the injection $\mathscr{I}_2$ is simply the identity; in order to see the effect of the injection $\mathscr{I}_1$ in this case, let $u = \{u_0, u_1, u_2, \ldots\}$ be a sequence in $S(\mathbb{R}^m)$, and, for each integer $i \geqslant 0$, let $u_{i,1}, \ldots, u_{i,m}$ be the components of the vector u_i. Then, letting $v = \mathscr{I}_1 u$, where $v = \{v_0, v_1, v_2, \ldots\}$, it is obtained that, for each integer $i \geqslant 0$, the components of the vector v_i are $v_{i,j} = u_{i,j}$ for $j = 1, \ldots, m$, and $v_{i,j} = 0$ for $j = m+1, \ldots, p$. To simplify the notation, $\mathscr{I}_1 u$ and $\mathscr{I}_2 y$ are abbreviated and denoted by u and y respectively. It can be seen that, when stabilizing the system Σ_γ using the configuration of Fig. 1, stabilization is in fact obtained for the original system Σ in Fig. 2. We prove this fact in the Corollary. (Note that in Fig. 2, γ is to be interpreted as $\gamma \mathscr{I}_1$, consistent with the notation convention.)

There are several simplifications that result when the system Σ_γ is used as the basic system to be stabilized, instead of the system Σ. One such simplification is the fact that Σ_γ is always injective when Σ is strictly causal, as will be seen later. This means that Σ_γ has a left inverse. Moreover, when the original system Σ is recursive, the left inverse of Σ_γ is very easy to compute. Indeed, assume that Σ has a recursive representation

$$y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu})$$

with $\eta \geqslant \mu$. Let $u \in S(\mathbb{R}^m)$ be an input sequence, and let $y := \Sigma u$ be the corresponding output sequence. Denoting $z := \Sigma_\gamma u$, and using the abbreviated notation given in the previous paragraph, $z = y + \gamma u$ is obtained, so that $z_i = y_i + \gamma u_i$ for all integers $i \geqslant 0$. Therefore,

$$\begin{aligned} z_{k+\eta+1} &= y_{k+\eta+1} + \gamma u_{k+\eta+1} \\ &= f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu}) + \gamma u_{k+\eta+1} \\ &= f((z-\gamma u)_k, \ldots, (z-\gamma u)_{k+\eta}, u_k, \ldots, u_{k+\mu}) + \gamma u_{k+\eta+1} \end{aligned}$$

and, invoking the invertibility of γ, we obtain

$$\left.\begin{aligned} &u_{k+\eta+1}=\gamma^{-1}\{z_{k+\eta+1}-f((z-\gamma u)_k,\ldots,(z-\gamma u)_{k+\eta},u_k,\ldots,u_{k+\mu})\}\\ &\quad k=0,1,2,\ldots\\ &u_i=\gamma^{-1}\{z_i-y_i\}\quad \text{for } i=0,\ldots,\eta \end{aligned}\right\}\tag{2.2}$$

where $y_0,\ldots,y_\eta$ are the given initial conditions of the system Σ, and where the relations are valid for any sequence $z\in\operatorname{Im}\Sigma_\gamma$. In view of the fact that $\mu\leqslant\eta$ by the strict causality of the given system Σ, it follows that the input sequence u of Σ_γ can readily be computed from the output sequence z of Σ_γ in a recursive manner, using the given initial conditions and recursion function of the system Σ. This evidently amounts to a left inversion of the system Σ_γ, and this procedure will be used repeatedly throughout the discussion. It is also clear from (2.2) that the left inverse is causal, and the discussion is summarized in the following.

Proposition 1

Let $\Sigma:S(\mathbb{R}^m)\to S(\mathbb{R}^q)$ be a strictly causal recursive system having a recursive representation $y_{k+\eta+1}=f(y_k,\ldots,y_{k+\eta},u_k,\ldots,u_{k+\mu})$ where $\eta\geqslant\mu$. Let $p:=\max\{m,q\}$ and let γ be a $p\times p$ constant invertible matrix. Then, the system $\Sigma_\gamma:S(\mathbb{R}^m)\to\operatorname{Im}\Sigma_\gamma$ defined by (2.1) is a bicausal system.

As mentioned in the Introduction, the discussion in this paper hinges to a large extent on the theory of fraction representations of non-linear systems developed by Hammer (1984 b, 1986, and in particular 1987), so a brief review of some of the basic aspects of this theory is now provided, starting with right fraction representations. First, it is remarked that the theory of fraction representations is only needed here for systems with bounded input spaces, so it is assumed that there is a fixed (but arbitrary) real number $\alpha>0$ such that all the systems have $S(\alpha^m)$ as their domain of input sequences. Now, let $\Sigma:S(\alpha^m)\to S(\mathbb{R}^p)$ be a non-linear system. A right fraction representation of the system Σ involves an integer $q>0$, a subspace $S\subset S(\mathbb{R}^q)$, and a pair of stable systems $P:S\to S(\mathbb{R}^p)$ and $Q:S\to S(\alpha^m)$, where Q is invertible, such that $\Sigma=PQ^{-1}$. The subspace S is called the 'factorization space' of the fraction representation. Of particular importance are coprime right fraction representations, which are fraction representations in which the systems P and Q are right coprime according to the following definition (Hammer 1985 a, 1987).

Definition 1

Let $S\subset S(\mathbb{R}^q)$ be a subspace. A pair of stable systems $P:S\to S(\mathbb{R}^p)$ and $Q:S\to S(\mathbb{R}^m)$ are *right coprime* if the following two conditions are satisfied.

(*a*) For every real $\tau>0$ there is a real $\theta>0$ such that

$$P^*[S(\tau P)]\cap Q^*[S(\tau^m)]\subset S(\theta^q)$$

(*b*) For every real $\tau>0$, the set $S\cap S(\tau^q)$ is a closed subset of $S(\tau^q)$.

For a discussion of the intuitive interpretation of this definition, see Hammer (1985 a, b). The concept of coprimeness is of fundamental importance to the theory of the stabilization of non-linear systems for the following reasons. Consider a non-linear system $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ having a right fraction representation $\Sigma = PQ^{-1}$. As seen in (1.4) and (1.5), in order to stabilize the system Σ through the control configuration in Fig. 1, a pair of stable systems A and B must be found for which the system $M := AP + BQ$ is unimodular. From the analogy of this situation with the situation in the theory of linear systems, it is expected that the existence of the stable systems A and B hinges on the coprimeness of the systems P and Q. Indeed, as the following theorem (taken from Hammer 1987) states, such stable systems A and B can always be found when P and Q are right coprime. Using obvious terminology, it is said that $\Sigma = PQ^{-1}$ is a right coprime fraction representation when the systems P and Q are right coprime.

Theorem 1

Let $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ be a system, and assume it has a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P: S \to S(\mathbb{R}^p)$ and $Q: S \to S(\alpha^m)$, and where $S \subset S(\mathbb{R}^q)$ for some integer $q > 0$. Then, for every unimodular system $M: S \to S$, there exists a pair of stable systems $A: S(\mathbb{R}^p) \to S(\mathbb{R}^q)$ and $B: S(\alpha^m) \to S(\mathbb{R}^q)$ such that $AP + BQ = M$.

The explicit construction of stable systems A and B satisfying the requirements of Theorem 1 is one of the main topics of this paper, and it will be discussed in detail in § 3. Now, the review of the theory of fraction representations is continued, turning to the question of their existence. Generally speaking, not every non-linear system possesses a right coprime fraction representation. However, as it turns out, most systems of practical interest do possess such representations. The existence of right coprime fraction representations is related in a fundamental way to the concept of a homogeneous system, which is defined as follows (Hammer 1985 a, 1987).

Definition 2

A system $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is a *homogeneous system* if the following holds for every real number $\alpha > 0$: for every subspace $S \subset S(\alpha^m)$ for which there exists a real number $\tau > 0$ satisfying $\Sigma[S] \subset S(\tau^p)$, the restriction of Σ to the closure $\bar{S}$ of S in $S(\alpha^m)$ is a continuous map $\Sigma: S \to S(\tau^p)$.

Intuitively speaking, a homogeneous system has the property that it exhibits continuity over sets of input sequences which generate bounded output sequences. The significance of homogeneous systems to the discussion comes from the fact that they are the only systems possessing right coprime fraction representations, as stated in the following theorem (Hammer 1985 a, 1987).

Theorem 2

An injective system $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ has a right coprime fraction representation if and only if it is a homogeneous system.

Homogeneous systems are quite common in engineering practice; in fact, as the next statement (reproduced from Hammer 1987) points out, most systems of practical interest are homogeneous.

Proposition 2

Let $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ be a recursive system. If Σ has a recursive representation

$y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu})$ with a continuous recursion function f, then Σ is a homogeneous system.

Thus, our attention can be confined to homogeneous systems without significantly restricting the applicability of the results in practice. It is also important to notice that construction of the system Σ_γ in (2.1) does not destroy homogeneity, as can be seen from the following statement (Hammer 1987).

Proposition 3

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a homogeneous system, and let Σ_γ be defined as in (2.1). Then $\Sigma_\gamma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is a homogeneous system.

As seen in Hammer (1987), it is rather easy to construct a right coprime fraction representation for an injective homogeneous system $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$. Indeed, let $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ be an injective homogeneous system. Then, since Σ is injective, its restriction $\Sigma: S(\alpha^m) \to \text{Im}\,\Sigma$ is a set isomorphism, and, consequently, it possesses an inverse $\Sigma^{-1}: \text{Im}\,\Sigma \to S(\alpha^m)$. As shown by Hammer (1987, § 3), Σ^{-1} is a stable system; hence defining the systems

$$\left.\begin{aligned} P &:= I: \text{Im}\,\Sigma \to \text{Im}\,\Sigma \\ Q &:= \Sigma^{-1}: \text{Im}\,\Sigma \to S(\alpha^m) \end{aligned}\right\} \tag{2.3}$$

a right fraction representation $\Sigma = PQ^{-1}$ is obtained, which can readily be seen to be right coprime. When there is one right coprime fraction representation $\Sigma = PQ^{-1}$ of the system $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$, then any other right coprime fraction representation of Σ is of the form $\Sigma = P_1 Q_1^{-1}$, where $P_1 = PM$ and $Q_1 = QM$, and where M is a unimodular transformation (Hammer 1985 a, 1987).

Another class of fraction representations that is important for the study of the problem of stabilizing a non-linear system is the class of left fraction representations. A left fraction representation of a non-linear system $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ involves an integer $q > 0$, a subspace $S \subset S(\mathbb{R}^q)$, and a pair of stable systems $G: \text{Im}\,\Sigma \to S$ and $T: S(\alpha^m) \to S$, where G is invertible, such that $\Sigma = G^{-1}T$. The main use of left fraction representations in this context is for the purpose of parametrizing the set of pairs of stable systems A, B which satisfy an equation of the form $AP + BQ = M$, where P and Q originate from a right coprime fraction representation $\Sigma = PQ^{-1}$, and where M is a fixed unimodular system. Indeed, let $S \subset S(\mathbb{R}^q)$ be the factorization space of the right coprime fraction representation $\Sigma = PQ^{-1}$, and let $\Sigma = G^{-1}T$ be a left fraction representation of the system Σ, having the factorization space $S' \subset S(\mathbb{R}^r)$. Assume we have one pair of stable systems $A: \text{Im}\,\Sigma \to S(R^q)$ and $B: S(\alpha^m) \to S(\mathbb{R}^q)$ satisfying $AP + BQ = M: S \to S$. To obtain other pairs of such systems, simply proceed as follows. Choose an arbitrary stable system $h: S(\mathbb{R}^r) \to S(\mathbb{R}^q)$, and define the stable systems

$$\left.\begin{aligned} A' &:= A - hG \\ B' &:= B + hT \end{aligned}\right\} \tag{2.4}$$

Then, using the fact that $PQ^{-1} = G^{-1}T$, or $GP = TQ$, obtain

$$A'P + B'Q = AP - hGP + BQ + hTQ = AP + BQ = M$$

and A', B' satisfy the equation. Thus, note that left fraction representations allow us to parametrize solutions of the basic equation in a rather transparent way. In fact, when $\Sigma : S(\alpha^m) \to S(\mathbb{R}^p)$ is a homogeneous system, all pairs of stable systems A', B' satisfying $A'P + B'Q = M$ can be obtained simply by varying the stable system h in (2.4) (Hammer 1987, § 4). The existence of left fraction representations for the systems considered is guaranteed by the following result, which is reproduced from Hammer (1987).

Theorem 3

An injective homogeneous system $\Sigma : S(\alpha^m) \to S(\mathbb{R}^p)$ has a left fraction representation.

It is also quite easy to construct a left fraction representation for an injective homogeneous system $\Sigma : S(\alpha^m) \to S(\mathbb{R}^p)$. Indeed, using the fact that $\Sigma^{-1} : \text{Im}\, \Sigma \to S(\alpha^m)$ is a stable system (Hammer 1987, § 3), and letting $I : S(\alpha^m) \to S(\alpha^m)$ be the identity map, the pair of stable systems

$$\left.\begin{aligned} G &:= \Sigma^{-1} : \text{Im}\, \Sigma \to S(\alpha^m) \\ T &:= I : S(\alpha^m) \to S(\alpha^m) \end{aligned}\right\} \tag{2.5}$$

induces a left fraction representation $\Sigma = G^{-1}T$ (Hammer 1987, § 4).

Internal stability becomes an issue of major concern whenever several systems are interconnected and combined into one composite system. Although a composite system may exhibit stable behaviour on the input/output relationship it induces, it may not retain this stability when interferences and noises disturb the input ports of the individual systems of which it consists. The notion of internal stability addresses this phenomenon, and, qualitatively, a composite system is said to be internally stable whenever such internal noises and interferences do not destroy the stability of the overall system. Of course, every closed-loop control system must be internally stable if it is to be of any practical use. In order to discuss the internal stability of the control configuration (Fig. 1), we now redraw the diagram, including all the possible noises, which are denoted by v_1, v_2, v_3 and v_4. The internal signals of the configuration are denoted by $w_1, \ldots, w_4$, and it is noted explicitly that the output sequence y depends on the noises $v_1, \ldots, v_4$ as well as the input sequence u.

The formal definition that will be used for the notion of internal stability is now stated.

Definition 3

Let $\theta > 0$ be a real number. The composite system of Fig. 3 is *internally stable* (for input sequences bounded by θ) if the following conditions are satisfied.

(*a*) The input/output map $\Sigma_{(\pi,\varphi)} : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is stable when restricted to $S(\theta^m)$.

(*b*) There exists a real $\beta > 0$ such that, for every input sequence $u \in S(\theta^m)$ and for all noise signals $|v_i| \leqslant \beta$, $i = 1, \ldots, 4$, one has

(i) for every real $\varepsilon > 0$ there is a real $\delta > 0$ such that, whenever

$$\rho(v_i) < \delta \quad \text{for all } i = 1, \ldots, 4$$

then

$$\rho[y(u, v_1, v_2, v_3, v_4) - y(u, 0, 0, 0, 0)] < \varepsilon$$

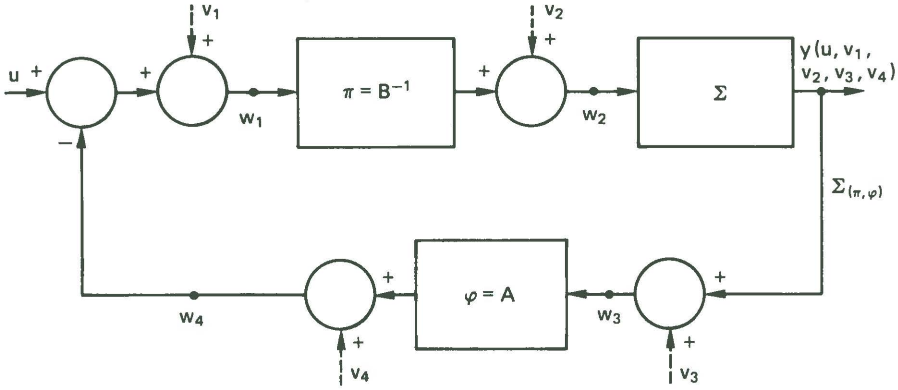

Figure 3.

(ii) there is a real $N > 0$ such that the internal signals satisfy $|w_i| \leqslant N$ for all $i = 1, \ldots, 4$.

Generally speaking, internal stability as defined in Definition 3 assures that the output signal y depends in a continuous way on the input signal u and on the noise signals v_1, v_2, v_3, v_4, as long as the amplitude of the input signal does not exceed θ and the amplitudes of the noise signals do not exceed β. In addition, through condition (*b*) (ii), internal stability also guarantees that the amplitudes of the internal signals $w_1, \ldots, w_4$ remain bounded by the fixed bound N throughout the entire permissible range of operation.

The task of guaranteeing that a certain composite system is internally stable is, in general, much more difficult than the task of making the composite system input/output stable. However, as seen shortly, one of the main advantages of using the configuration in Fig. 1 with the compensators π and φ chosen according to (1.2) is that, for the resulting configuration, input/output stability almost automatically assures internal stability as well. Before discussing this point, the following type of systems are introduced (Hammer 1986).

Definition 4

Let $A: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a stable system, and let $\theta > 0$ be a real number. We say that the system A is *differentially bounded* by θ if there exists a real $\varepsilon > 0$ such that, for every pair of elements y, $y' \in S(\mathbb{R}^m)$ satisfying $|y - y'| < \varepsilon$, one has $|A(y) - A(y')| < \theta$.

The property of being differentially bounded assures that a stable system, which is by definition continuous with respect to ρ, also exhibits 'nice' behaviour with respect to the l^∞ norm. For instance, it is easy to see that every uniformly l^∞-continuous system is differentially bounded by θ, for any real $\theta > 0$. As another simple example of a stable and differentially bounded system, consider the following. Let $\sigma: S(\mathbb{R}^m) \to S(1^m)$ be any bounded stable system with amplitude of output sequences bounded, say, by 1. Then the system $A := I + \sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$, where $I: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ denotes the identity system, is evidently stable and differentially bounded by $(1 + \delta)$, for any real $\delta > 0$.

The internal stability properties of the control configuration in Fig. 1, with compensators π and φ chosen in accordance with (1.2), are now discussed. Let $\Sigma = PQ^{-1}$ be a right coprime fraction representation of the given system Σ. As seen in the discussion following (1.2), the closed-loop system $\Sigma_{(\pi,\varphi)}$ can be made input/output stable by choosing the stable systems A and B so that the system $M = AP + BQ$ is unimodular. As can be seen from the next statement, which was proved by Hammer (1986, § 3), this will in fact make $\Sigma_{(\pi,\varphi)}$ internally stable whenever A and B are also differentially bounded. (The other assumptions of Theorem 4 are simply related to the description of M, and they will be discussed further in § 3.)

Theorem 4

Let $\Sigma : S(\alpha^m) \to S(\mathbb{R}^p)$ be a causal homogeneous system, and let $\theta > 0$ be a real number. Let $\Sigma = PQ^{-1}$ be a right coprime fraction representation, and let $S \subset S(\mathbb{R}^q)$ be its factorization space. Assume S contains a subset S' which is stability-morphic to $S((5\theta)^m)$, and let $M : S' \to S((5\theta)^m)$ be a unimodular transformation. Assume there is a pair of stable systems $A : S(\mathbb{R}^p) \to S(\mathbb{R}^m)$ and $B : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ satisfying the equation $APv + BQv = Mv$ for all $v \in S'$, where A is causal and B is bicausal. If A and B are differentially bounded by θ, then the closed-loop system $\Sigma_{(B^{-1},A)}$ is internally stable for all input sequences $u \in S(\theta^m)$.

Let $\Sigma : S(\alpha^m) \to S(\mathbb{R}^q)$ be a strictly causal homogeneous system, and let $\Sigma_\gamma : S(\alpha^m) \to S(\mathbb{R}^p)$ be the system constructed from it in (2.1). The system Σ_γ, being a sum of systems, is of course a composite system. Nevertheless, ignore this fact for a short while, and regard Σ_γ as a single system, so that Theorem 4 can be applied to it. In consistency with our notation, the system that results when Σ_γ is inserted for Σ in the configuration in Fig. 1 is denoted by $\Sigma_{\gamma(\pi,\varphi)}$. It is assumed that the system Σ_γ satisfies the conditions of Theorem 4, and let $\Sigma_\gamma = PQ^{-1}$ be a right coprime fraction representation of Σ_γ for which the unimodular transformation M described in Theorem 4 exists. Using the notational convention of the paragraph following (2.1) to omit the injections $\mathscr{I}_1$ and $\mathscr{I}_2$, consider the system $P_\sigma := \Sigma Q$. Since $\Sigma = \Sigma_\gamma - \gamma$, then

$$P_\sigma = \Sigma_\gamma Q - \gamma Q = P - \gamma Q \tag{2.6}$$

and it follows that P_σ is stable and $\Sigma = P_\sigma Q^{-1}$ is a right fraction representation. Moreover, it is easy to see that, since $\Sigma_\gamma = PQ^{-1}$ is a right coprime fraction representation, so also is $\Sigma = P_\sigma Q^{-1}$. Finally, let A and B be systems satisfying the conditions of Theorem 4, so that when Σ_γ is regarded as a single system as opposed to the composite system it really is, the closed-loop system $\Sigma_{\gamma(B^{-1},A)}$ is internally stable. Referring now to the control configuration in Fig. 2, it is easy to see by direct computation that

$$\Sigma_{(\gamma,B^{-1},A)} = \Sigma_{\gamma(B^{-1},A)} - \gamma Q M^{-1} = (P - \gamma Q)M^{-1} = P_\sigma M^{-1} \tag{2.7}$$

which implies that $\Sigma_{(\gamma,B^{-1},A)}$ is input/output stable whenever the conditions of Theorem 4 are satisfied for Σ_γ. (Familiarity with the proof of Theorem 4, given by Hammer 1987, § 3, may be helpful for the remaining part of this paragraph.) Further, in order to investigate the internal stability of the configuration in Fig. 2, the effect of possible noises added at the points (a), (b) (c) (d) and (e) and the boundedness of the internal signals at these points must be considered, in addition to the noises and internal signals that are considered for the configuration $\Sigma_{\gamma(B^{-1},A)}$. For this purpose, notice that when Σ_γ is inserted for Σ in Fig. 3, the points w_2 and w_3 in that diagram become points w_2 and w_3 in Fig. 2, and the output y of Fig. 3 becomes the output z of Fig. 2. Consequently, when the conditions of Theorem 4 are satisfied for Σ_γ, the internal signals w_2 and w_3 in Fig. 2 are bounded. In view of the fact that γ is a constant matrix, this evidently implies that the internal signals at the points (a), (b), (c) and y in Fig. 2 are bounded as well. Also, the output z in Fig. 2 depends continuously on the internal noises $v_1, \ldots, v_4$ of Fig. 3 when Σ_γ is inserted for Σ and the conditions of Theorem 4 hold for Σ_γ. Noticing that $y = z - \gamma\Sigma_\gamma^{-1}z$, and recalling that the homogeneity of Σ_γ together with the boundedness of the internal signals imply that the restriction of Σ_γ^{-1} to the output space at z is stable (see (2.3)), the fact that y depends continuously on the internal noises $v_1, \ldots, v_4$ under the given circumstances is obtained. Further, in order to consider the effect of the new noises, note that noises added at the points (b), (c) and (e) in Fig. 2 can be represented as equivalent noises added at the point (d) in that diagram, and therefore can be equivalently represented in the noise v_3 of the configuration in Fig. 3 for Σ_γ. Consequently, using the fact that γ is a constant matrix, it is easy to see that when the conditions of Theorem 4 hold for Σ_γ, the signal z of Fig. 2 depends continuously on the internal noises at the points (b), (c) and (e) so that, in view of our previous observations, y also depends continuously on these noises. Finally, the effect of a noise at the point (a) must be considered. A slight reflection shows that a noise n added at the point (a) can be represented as the combination of a noise n added at the point (g) together with a noise $-n$ added at the point (b) in Fig. 2. Therefore, in view of the discussion earlier in this paragraph, y depends continuously on the noise n whenever the conditions of Theorem 4 hold for Σ_γ. In summary; whenever the conditions of Theorem 4 are satisfied for Σ_γ, the closed-loop configuration $\Sigma_{(\gamma,B^{-1},A)}$ will in fact be internally stable. This conclusion is stated in the following corollary.

Corollary

Let $\Sigma:S(\alpha^m) \to S(\mathbb{R}^q)$ be a strictly causal homogeneous system, let $\Sigma_\gamma:S(\alpha^m) \to S(\mathbb{R}^p)$ be given by (2.1), and let $\theta > 0$ be a real number. Let $\Sigma_\gamma = PQ^{-1}$ be a right coprime fraction representation, and let $S \subset S(\mathbb{R}^r)$ be its factorization space. Assume S contains a subset S' which is stability-morphic to $S((5\theta)^m)$, and let $M: S' \to S(5\theta)^m)$ be a unimodular transformation. Assume there is a pair of stable systems $A: S(\mathbb{R}^p) \to S(\mathbb{R}^m)$ and $B: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ satisfying the equation $APv + BQv = Mv$ for all $v \in S'$, where A is causal and B is bicausal. If A and B are differentially bounded by θ, then the closed-loop system $\Sigma_{(\gamma,B^{-1},A)}$ is internally stable for all input sequences $u \in S(\theta^m)$.

The Corollary forms the basis for the procedure of stabilization and dynamics assignment developed in this paper. In the next section, the unimodular transformation M mentioned in the Corollary is discussed, and it is shown how systems A and B satisfying the conditions of the Corollary can actually be constructed as implementable combinations of recursive systems.

3. Dynamics assignment

In this section the theory of dynamics assignment for non-linear recursive systems is developed. Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a recursive system having a recursive representation of the form (1.1). Using the basic theoretical framework reviewed in the previous section, compensators that stabilize the system Σ are derived and the desired dynamical behaviour to the stabilized closed-loop system is assigned. The stabilizing compensators obtained consist of recursive systems, and thus can be implemented on digital computers. In this way, the results become directly applicable to practical situations. First consider a particular case, which will allow us to demonstrate in a transparent way the basic ideas on which the theory is based. Specifically, assume that the system Σ whose dynamical behaviour is to be altered has an output space which is of the same dimension as its input space, namely, that $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$. It will be seen later that this restriction is inconsequential, and that it can readily be removed without substantially changing the state of affairs.

So, let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be a strictly causal homogeneous and recursive system. Following the discussion (2.1), define the system

$$\Sigma_\gamma := \gamma I + \Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m) \tag{3.1}$$

where $I : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ is the identity system, and where γ is an $m \times m$ constant invertible matrix, whose significance will be discussed later. Clearly, Σ_γ depends on the matrix γ. It can readily be verified that, in this particular case, the image of Σ_γ is equal to the whole space $S(\mathbb{R}^m)$. As seen in Proposition 1, the strict causality of Σ combined with the invertibility of the matrix γ, imply that the system Σ_γ is bicausal. Consequently, Σ_γ induces a set isomorphism of $S(\mathbb{R}^m)$ onto $S(\mathbb{R}^m)$, and thus the set $S(\delta^m)$ is contained in $\operatorname{Im} \Sigma_\gamma$ for every real $\delta > 0$. Now, assume that there is a real number $\delta > 0$ such that all the output sequences in $S(\delta^m)$ originate from bounded input sequences. In formal terms, this assumption is stated as follows.

Condition 1

There is an $m \times m$ non-singular constant matrix γ and a real number $\delta > 0$ such that $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for some real $\alpha > 0$.

Now, since the system Σ is homogeneous, it follows from Proposition 3 that the system $\Sigma_\gamma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ is homogeneous as well. When Condition 1 holds, this implies, as shown in Lemma 1, that the restriction of Σ_γ^{-1} to $S(\delta^m)$ is stable.

Lemma 1

Let $D : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be an injective homogeneous system, and let $\alpha > 0$ be a real number. Then, the restriction $D^{-1} : D[S(\alpha^m)] \to S(\alpha^m)$ is a stable system.

Proof

Denote $S := D[S(\alpha^m)]$, let $\delta > 0$ be a real number, and let $S_\delta := D^{-1}[S \cap S(\delta^m)] \subset S(\alpha^m)$. Now, since $D[S_\delta] \subset S(\delta^m)$ and the system D is homogeneous, it follows that the restriction $D : \bar{S}_\delta \to S(\delta^m)$, where $\bar{S}_\delta$ is the closure of S_δ in $S(\alpha^m)$, is a continuous map. Since $\bar{S}_\delta$ is a compact set and D is injective, this implies that the restriction $D : \bar{S}_\delta \to D[\bar{S}_\delta]$ is a homeomorphism (see, e.g. Kuratowski 1961), and, consequently, the restriction of D^{-1} to $D[\bar{S}_\delta]$ is continuous. Noticing that

$$S \cap S(\delta^m) \subset D[\bar{S}_\delta]$$

it is found that the restriction of D^{-1} to $S \cap S(\delta^m)$ is a continuous map, and

$$D^{-1}[S \cap S(\delta^m)] \subset D^{-1}[S] = S(\alpha^m)$$

Finally, since this is true for any real $\delta > 0$, it follows that the restriction of D^{-1} to S is stable, and our proof is concluded. □

Assume now that the system Σ_γ satisfies Condition 1. Then, as seen from Lemma 1, the restriction $\Sigma_\gamma^{-1}:\Sigma_\gamma[S(\alpha^m)]\to S(\alpha^m)$ is stable. Recalling that Σ_γ was bicausal, it is found that the system $Q:=\Sigma_\gamma^{-1}:\Sigma_\gamma[S(\alpha^m)]\to S(\alpha^m)$ is bicausal and stable. Using the given recursion function f of the system Σ, Q can be computed in a straightforward manner using (2.2). Let $P:=I_1:\Sigma[S(\alpha^m)]\to\Sigma[S(\alpha^m)]$ denote the identity system. Then, in view of (2.3), a right coprime fraction representation $\Sigma_\gamma = PQ^{-1}$ is obtained, which is valid over the subspace $S(\alpha^m)$. Note that the factorization space for this fraction representation is $S:=\Sigma[S(\alpha^m)]\subset S(\mathbb{R}^m)$ and that, in view of the Condition, $S(\delta^m)\subset S$ is true. Next, the restriction $Q:S(\delta^m)\to S(\alpha^m)$ is extended into a stable, bounded, and causal system $Q_*:S(\mathbb{R}^m)\to S(\alpha^m)$ which consists of a combination of recursive systems. This can readily be done by setting

$$Q_* := QE \tag{3.2}$$

where $E:S(\mathbb{R}^m)\to S(\zeta^m)$ is the static extension of the identity system constructed in Lemma 2, with $\zeta<\delta$. (Choosing $\zeta<\delta$ ensures that small noises appearing between Q and E will not deviate from the domain of definition of Q, as long as their amplitude does not exceed $\delta-\zeta$.)

Lemma 2

Let $\zeta>0$ be a real number, and let $I_\zeta:S(\zeta^m)\to S(\zeta^m)$ be the identity system. There is a recursive, causal, stable, and unformly l^∞-continuous system $E:S(\mathbb{R}^m)\to S(\zeta^m)$ which is an extension of I_ζ.

Proof

First define a function $e:\mathbb{R}^m\to[-\zeta,\zeta]^m$ as follows. For every vector $x=(x_1,\ldots,x_m)\in\mathbb{R}^m$ set $e(x_1,\ldots,x_m):=(\alpha_1,\ldots,\alpha_m)$ where $\alpha_i:=x_i$ if $|x_i|\leqslant\zeta$ and $\alpha_i:=\zeta$ sign (x_i) if $|x_i|>\zeta$, and where sign $(\cdot)$ is ± 1, depending on the sign of the argument. Then, define the system $E:S(\mathbb{R}^m)\to S(\zeta^m)$ as the recursive system with the representation

$$E: y_k := e(u_k), \quad k=1,2,\ldots \tag{3.3}$$

where $y=Eu$. This system, which is a static system, clearly satisfies the requirements of Lemma 2. □

Stabilizing compensators can now be constructed for a given strictly causal homogeneous system $\Sigma:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$ that satisfies Condition 1. At first, the objective is limited to merely stabilizing the system Σ, and the problem of dynamics assignment is considered later (see Theorem 6). Compensators π and φ of the form (1.2) are used, and the systems A and B which determine them are constructed. Of course, there is a whole class of systems A and B that will yield suitable stabilizing compensators π and φ. However, as seen in (2.4), once one suitable solution for the systems A and B is available, a whole class of such solutions can be derived in a very simple way. Thus, begin by finding one particular solution A and B, and then use it to obtain other solutions. Let $\theta>0$ be the specified amplitude of the input sequences to the final closed-loop system $\Sigma_{(\gamma,\pi,\varphi)}$, let $\delta>0$ be as in Condition 1, choose a number

$0 < \zeta < \delta$, and denote $\beta := 5\theta/\zeta$. Then, the system $M := \beta I : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ being a non-zero multiple of the identity system, is unimodular, and it satisfies $M[S(\zeta^m)] = S((5\theta)^m)$. Thus, taking $S = \Sigma[S(\alpha^m)]$ and $S' = S(\zeta^m)$, it can be seen that this unimodular transformation M satisfies the conditions of the Corollary. Now, let $\varepsilon > 0$ be a fixed number, and define the stable systems

$$\left.\begin{aligned} A &:= \beta I - \varepsilon Q_* : S(\mathbb{R}^m) \to S(\mathbb{R}^m) \\ B &:= \varepsilon I : S(\mathbb{R}^m) \to S(\mathbb{R}^m) \end{aligned}\right\} \tag{3.4}$$

where $I : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ is the identity system, and Q_* is given by (3.2). Notice that with this choice, B is evidently uniformly l^∞-continuous, and thus is differentially bounded by θ, for any value of $\theta > 0$. The precompensator $\pi = B^{-1} = (1/\varepsilon)I$ is, for this choice of B, just a simple amplifier with an amplification factor of $1/\varepsilon$. Moreover, by choosing $\varepsilon < \theta/\alpha$, and recalling that the output sequences of Q_* are bounded by α, it is found that A is differentially bounded by θ. It is also clear that B is bicausal and, since Q_* is causal, A is causal. Finally, recalling the construction of the right coprime fraction representation $\Sigma_\gamma = PQ^{-1}$ with $Q := \Sigma_\gamma^{-1} : S \to S(\alpha^m)$ and $P := I_1$, and using the notation of the Corollary with $S' := S(\zeta^m)$ and $M := \beta I$, it is found, for every element $v \in S(\zeta^m)$, that $APv + BQv = (\beta I - \varepsilon Q)I_1 v + \varepsilon Qv = (\beta Iv - \varepsilon Qv) + \varepsilon Qv = \beta Iv$, and it follows that the conditions of the Corollary are satisfied. Consequently, letting $\pi := B^{-1}$ and $\varphi := A$, the closed-loop system $\Sigma_{(\gamma,\pi,\varphi)}$ becomes internally stable for all input sequences from $S(\theta^m)$. Moreover, since Q_* consists of recursive systems, it follows directly from (3.4) that the stabilizing compensators π and φ consist of recursive systems. As can be seen from their explicit representations, these compensators are also rather easy to implement. The discussion up to this point is summarized in the following theorem.

Theorem 5

Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be a strictly causal homogeneous system. Let Σ_γ be given by (3.1), and let $\theta > 0$ be a real number. Assume there is an $m \times m$ constant non-singular matrix γ and a real number $\delta > 0$ such that

$$S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$$

for some real $\alpha > 0$. Let $\zeta, \beta, \varepsilon > 0$ be real numbers satisfying $\zeta < \delta$, $\beta \geqslant 5\theta/\zeta$, and $\varepsilon < \theta/\alpha$. Then, for the compensators $\pi := B^{-1}$ and $\varphi := A$, where A and B are given by (3.4) and Q_* is given by (3.2), and the closed-loop system $\Sigma_{(\gamma,\pi,\varphi)}$ is internally stable for all input sequences bounded by θ.

One important advantage obtained by using the control configuration in Fig. 2 with the compensators given by (3.4) is that the precompensator $\pi = (1/\varepsilon)I$ can be chosen to have arbitrarily high gain, since, as seen from Theorem 5, the positive number ε can be chosen arbitrarily small. As is widely known in control-theoretic folklore, high forward gain is extremely effective in dealing with uncertainties in the description of the given system Σ. Thus it can be expected that, by appropriately choosing the value of ε, the stability of the closed-loop configuration can be maintained despite variations in the system Σ. This point will be discussed in greater detail in a separate report.

As seen from Theorem 5 and the discussion leading to it, it is rather easy to stabilize a non-linear recursive system Σ when a constant matrix γ satisfying

Condition 1 is known. As will be seen later, the matrix γ exists under quite general conditions, and its computation can be reduced to the solution of a standard problem in control theory, for which computational algorithms are well known. Since the main interest here is in recursive systems, it would be useful to see first what the implications of Condition 1 are in terms of the recursion function f of the system Σ that has to be stabilized. As mentioned in the Introduction to the paper, the discussion will be restricted to non-linear recursive systems $\Sigma: S(\mathbb{R}^m) \rightarrow S(\mathbb{R}^p)$ given in terms of state equations of the form

$$x_{k+1} = f(x_k, u_k)$$

where $f: \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is a continuous function, and the initial condition x_0 is given. In this preliminary stage of the discussion, it is assumed, as above, that $p = m$. It will be seen later that the situation in the general case $p \neq m$ is very similar. The aim is to find properties of the function f that guarantee that Condition 1 is satisfied. One such property, which is shared by a large class of practical systems, is described in the following proposition.

Proposition 4

Let $\Sigma: S(\mathbb{R}^m) \rightarrow S(\mathbb{R}^m)$ be a recursive system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where $f: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a continuous function, and the initial condition is $x_0 = 0$. Assume that there is an $m \times m$ constant non-singular matrix γ for which the following holds. There are real numbers $\delta > 0$, $c \geqslant 0$ and $0 \leqslant n(\gamma) < 1$ such that

$$|f((v - \gamma u), u)| \leqslant n(\gamma)|\gamma u| + c \quad \text{for all } |v| \leqslant \delta$$

Then the system Σ satisfies Condition 1.

Proof

Let v be an arbitrary sequence in $S(\delta^m)$ and let $u := \Sigma_\gamma^{-1} v$. Then, using (2.2) and the fact that $x_0 = 0$,

$$\gamma u_{k+1} = v_{k+1} - f((v_k - \gamma u_k), u_k), \quad \gamma u_0 = v_0 - x_0 = v_0$$

and

$$|\gamma u_{k+1}| \leqslant |v_{k+1}| - |f((v_k - \gamma u_k), u_k)|$$

Using the assumption of Proposition 4 and the fact that $v \in S(\delta^m)$, so that $|v_{k+1}| \leqslant \delta$, it is found that

$$|\gamma u_{k+1}| \leqslant n(\gamma)|\gamma u_k| + (\delta + c) \tag{3.5}$$

Using these assumptions again, an $m \times m$ constant non-singular matrix γ is now chosen for which $0 \leqslant n(\gamma) < 1$. Since

$$|\gamma u_0| \leqslant |v_0| \leqslant \delta \quad \text{and} \quad |n(\gamma)| < 1$$

then inequality (3.5) implies that the sequence of norms $\{|\gamma u_k|\}_{k=0}^{\infty}$ is bounded for any sequence $v \in S(\delta^m)$. By the invertibility of γ, this implies that the sequence of norms $\{|u_k|\}_{k=0}^{\infty}$ is bounded as well, e.g. $|u_k| \leqslant \alpha$ for all integers $k \geqslant 0$ and for all $v \in S(\delta^m)$.

Thus, there is a real number $\alpha > 0$ for which

$$\Sigma_\gamma^{-1}[S(\delta^m)] \subset S(\alpha^m)$$

or

$$S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$$

and the Condition holds for the system Σ. □

In order to show how the conditions of Proposition 4 can be verified in practical situations, begin by considering their application to the case of single-input single-output systems. The situation in this case is very simple, and it provides insight into the situation in general. It will be seen later that analogous results are also valid in the multivariable case. Consider first the class of systems for which the non-linearity of the recursion function f is bounded.

Proposition 5

Let $\Sigma : S(\mathbb{R}) \to S(\mathbb{R})$ be a recursive system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the initial condition is $x_0 = 0$. Assume that the recursion function f is of the form $f(x, u) = ax + bu + \psi(x, u)$ where a and b are real numbers with $b \neq 0$, and where $\psi: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, say $|\psi| \leqslant N$ for some real number $N > 0$. Then the system Σ satisfies Condition 1 for any real $\delta > 0$.

Proof

Fix some real number $\delta > 0$, and consider all elements v with $|v| \leqslant \delta$. Using the notation of Proposition 4,

$$\begin{aligned} f(v - \gamma u, u) &= a(v - \gamma u) + bu + \psi(v - \gamma u, u) \\ &= \left(\frac{b}{\gamma} - a\right)\gamma u + av + \psi(v - \gamma u, u) \end{aligned}$$

so that

$$|f(v - \gamma u, u)| \leqslant \left|\left(\frac{b}{\gamma} - a\right)\right| |\gamma u| + |av + \psi(v - \gamma u, u)|$$

Letting $n(\gamma) := |b/\gamma - a|$ and $c := |a|\delta + N$, it is found that $|av + \psi(v - \gamma u, u| \leqslant c$ and $|f(v - \gamma u, u)| \leqslant n(\gamma)|\gamma u| + c$. Furthermore, since $b \neq 0$, it is also clear that there is a real number $\gamma \neq 0$ for which $n(\gamma) = |(b/\gamma - a)| < 1$. Thus the conditions of Proposition 4 are satisfied for any real $\delta > 0$. □

Another class of systems for which the conditions of Proposition 4 are easy to verify is the class of recursive systems for which the recursion function f is linearly bounded, in the following sense.

Proposition 6

Let $\Sigma : S(\mathbb{R}) \to S(\mathbb{R})$ be a recursive system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, with the initial condition $x_0 = 0$. Assume the recursion function f satisfies the following. There are real numbers a and b, with $b \neq 0$, such that $|f(x, u)| \leqslant |ax + bu|$ for all elements $x, u \in \mathbb{R}$. Then, the system Σ satisfies Condition 1 for any real $\delta > 0$.

Proof

The notation of Proposition 4 and its proof are used. Having $|f((v - \gamma u), u)| \leqslant |a(v - \gamma u) + bu|$ so that $|f((v - \gamma u), u)| \leqslant |(b/\gamma - a)|\,|\gamma u| + |av|$, let $\delta > 0$ be any real number, and consider all elements $v \in \mathbb{R}$ satisfying $|v| \leqslant \delta$. Setting $c := |a|\delta$ and $n(\gamma) := |(b/\gamma - a)|$, it follows that $|f(v - \gamma u), u)| \leqslant n(\gamma)|\gamma u| + c$ for all $|v| \leqslant \delta$. Moreover, since $b \neq 0$ there is a number $\gamma \neq 0$ for which $n(\gamma) < 1$. Thus, the conditions of Proposition 4 hold for any real $\delta > 0$. □

In order to consider the validity of Condition 1 for the case of systems Σ with more general types of recursion functions, the following refined version of Proposition 4 is stated.

Proposition 7

Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be a recursive system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$ where $f: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function, and the initial condition is $x_0 = 0$. Assume there is an $m \times m$ constant non-singular matrix γ for which the following holds. There are real numbers $\delta > 0$, $\kappa > 0$, $c \geqslant 0$, and $0 \leqslant n(\gamma) < 1$, such that

(*a*) $|f((v - \gamma u), u)| \leqslant n(\gamma)|\gamma u| + c$ whenever $|v| \leqslant \delta$ and $|\gamma u| \leqslant \kappa$, and

(*b*) $\dfrac{(\delta + c)}{(1 - n(\gamma))} + \delta \leqslant \kappa$

Then, the system Σ satisfies Condition 1.

Proof

First, since $\gamma u_k = v_k - x_k$ for all integers $k \geqslant 0$, and $x_0 = 0$, then $\gamma u_0 = v_0$, so that $|\gamma u_0| \leqslant \delta$. As seen in (3.5), $|\gamma u_{k+1}| \leqslant n(\gamma)|\gamma u_k| + (\delta + c)$ so that, by computing the recursion with $|n(\gamma)| < 1$, it follows that

$$|\gamma u_{k+1}| < (\delta + c)\left\{\sum_{n=0}^{\infty} [n(\gamma)]^n\right\} + \delta = \frac{(\delta + c)}{(1 - n(\gamma))} + \delta$$

Consequently, if $(\delta + c)/(1 - n(\gamma)) + \delta \leqslant \kappa$, then $|\gamma u_k| \leqslant \kappa$ for all integers $k \geqslant 0$, which implies that the restriction $|\gamma u| \leqslant \kappa$ in condition (*a*) of Proposition 7 is satisfied for all $v \in S(\delta^m)$. Therefore, as long as $v \in S(\delta^m)$, the conditions of Proposition 4 hold, and the assertion follows. □

As with Proposition 6, it will first be shown how the conditions of Proposition 7 can be applied to single-input single-output systems. Using this proposition, it is shown in the next statement that Condition 1 is valid for a rather large class of single-input single-output systems, basically consisting of all systems which have a differentiable recursion function in which the dependence on the input variable u is not degenerate. It will be seen later that an analogous result is also true in the multivariable case.

Proposition 8

Let $\Sigma: S(\mathbb{R}) \to S(\mathbb{R})$ be a recursive system, having a recursive representation $x_{k+1} = f(x_k, u_k)$, where $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and the initial condition is $x_0 = 0$. Assume that $f(0, 0) = 0$, and that, at the origin, f is differentiable and $\partial f / \partial u \neq 0$. Then, the system Σ satisfies Condition 1.

Proof

Since the function f is differentiable at the origin, it follows that

$$f(x, u) = ax + bu + \psi(x, u)$$

where

$$a := \frac{\partial f}{\partial x}(0, 0), \quad b := \frac{\partial f}{\partial u}(0, 0)$$

and where

$$\lim_{|(x,u)| \to 0} \frac{\psi(x, u)}{|(x, u)|} = 0$$

Consequently

$$f((v_k - \gamma u_k), u_k) = a(v_k - \gamma u_k) + bu_k + \psi = \left(\frac{b}{\gamma} - a\right)\gamma u_k + av_k + \psi$$

so that

$$|f((v_k - \gamma u_k), u_k)| \leqslant \left|\left(\frac{b}{\gamma} - a\right)\right| |\gamma u_k| + |a|\,|v_k| + |\psi| \leqslant n(\gamma)\,|\gamma u_k| + c$$

where

$$c := |a|\delta + |\psi| \quad \text{and} \quad n(\gamma) := \left|\left(\frac{b}{\gamma} - a\right)\right|$$

Recalling that $b \neq 0$ by the Proposition assumption, it follows that there is a real number $\gamma \neq 0$ for which $n(\gamma) < 1$; hence, one such value may be chosen for γ. For this value of γ, consider the inequalities

$$\frac{(\delta + c)}{(1 - n(\gamma))} + \delta \leqslant \kappa \quad \text{and} \quad |\gamma u| \leqslant \kappa$$

of Proposition 7. Substituting the value of c into the first inequality, it follows that

$$\frac{(\delta + |a|\delta + |\psi|)}{(1 - n(\gamma))} + \delta = \left[1 + \frac{(1 + |a|)}{(1 - n(\gamma))}\right]\delta + \frac{|\psi|}{(1 - n(\gamma))} \leqslant \kappa$$

Denote $\gamma_* := \max\{1, 1/|\gamma|\}$. Now, in view of the fact that

$$\lim_{|(x,u)| \to 0} \frac{\psi(x, u)}{|(x, u)|} = 0$$

there is a real number $\xi > 0$ for which

$$|\psi| < \frac{(1 - n(\gamma))\xi}{4\gamma_*}$$

for all pairs (x, u) satisfying $|(x, u)| < \xi$. Choose a number $\delta > 0$ so that

$$\left[1 + \frac{(1 + |a|)}{(1 - n(\gamma))}\right]\delta < \frac{\xi}{4\gamma_*}$$

which also implies that $\delta < \xi/4\gamma_*$. Using the numbers ξ and δ so selected, and taking $\kappa := \xi/2\gamma_*$, then

$$\frac{(\delta + c)}{(1 - n(\gamma))} + \delta \leqslant \frac{\xi}{4\gamma_*} + \frac{\xi}{4\gamma_*} = \frac{\xi}{2\gamma_*} = \kappa$$

so that condition (*b*) of Proposition 7 holds whenever $|(x. u)| < \xi$. Thus, it only remains to show that when $|\gamma u| \leqslant \kappa$ and $|v| \leqslant \delta$ (condition (*a*) of Proposition 7), the inequality $|(x, u)| < \xi$ holds. But $|\gamma u| \leqslant \kappa$ implies $|u| \leqslant \kappa/|\gamma| \leqslant \xi/2$. Also, since $v = x + \gamma u$, it follows that $x = v - \gamma u$, or

$$|x| \leqslant |v| + |\gamma u| \leqslant \delta + \kappa \leqslant \frac{\xi}{4\gamma_*} + \frac{\xi}{2\gamma_*} = \frac{3\xi}{4\gamma_*} < \frac{\xi}{\gamma_*} \leqslant \xi$$

Consequently, $|(x, u)| = \max\{|x|, |u|\} < \xi$. □

As the next step in the discussion, it is shown that results analogous to those described in Propositions 5, 6 and 8 are also valid in the case where $m > 1$. Only the generalizations of Propositions 5 and 8 are stated formally.

Proposition 9

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be a recursive system having a recursive representation of the form $x_{k+1} = f\{x_k, u_k)$, where $f: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and the initial condition is $x_0 = 0$. Assume that the recursion function f is of the form $f(x, u) = Fx + Gu + \psi(x, u)$, where the pair of $m \times m$ matrices (F, G) is controllable, and where $\psi: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a bounded continuous function, say $|\psi| \leqslant N$ for some real number $N > 0$. Then the system Σ satisfies Condition 1 for any real $\delta > 0$.

Proof

(For an $m \times m$ matrix A, a norm of A is denoted by $|A|$, such that $|Au| \leqslant |A|\,|u|$ for all elements $u \in \mathbb{R}^m$.) Let γ be a non-singular $m \times m$ matrix, let $\delta > 0$ be a real number, and let v be an arbitrary sequence in $S(\delta^m)$. Denote $u := \Sigma_\gamma^{-1} v$. Then, using the relations (2.2) for Σ_γ^{-1}, and substituting the particular form of f, it follows that

$$\begin{aligned}\gamma u_{k+1} &= v_{k+1} - f((v_k - \gamma u_k), u_k) = v_{k+1} - Fv_k + F\gamma u_k - Gu_k - \psi \\ &= (F - G\gamma^{-1})(\gamma u_k) + v_{k+1} - Fv_k - \psi\end{aligned}$$

Denoting $z := \gamma u$, $K := \gamma^{-1}$, and $w_k := v_{k+1} - Fv_k - \psi$, then

$$z_{k+1} = (F - GK)z_k + w_k \tag{3.6}$$

which can be regarded as the state representation of a linear system with state z and input w, operating under the static state feedback K. Now, since the pair (F, G) is controllable by the assumption above, there is a feedback matrix K (which is $m \times m$ here) such that all the eigenvalues of the matrix $(F - GK)$ have absolute value strictly less than 1 (Wonham 1967). Moreover, since an arbitrarily small change in K can transform it into a non-singular matrix in case it is singular, it is easy to see that the $m \times m$ matrix K can be chosen so that it is non-singular. Now, for all $v \in S(\delta^m)$ it is the case that

$$|w| \leqslant |v| + |Fv| + |\psi| \leqslant \delta + |F|\delta + N \leqslant v$$

for an appropriate real number $v > 0$. Also, the initial condition for the system (3.6) satisfies

$$|z_0| = |\gamma u_0| = |v_0 - x_0| = |v_0| \leqslant \delta$$

since $x_0 = 0$. Consequently, whenever $v \in S(\delta^m)$, the system (3.6) is operated from bounded initial conditions by bounded input sequences, and, having all its eigenvalues with absolute value strictly less than 1, it follows that there is a real number $N' > 0$ such that $|z| \leqslant N'$. Now, letting $\gamma := K^{-1}$ it follows that

$$|\gamma u| \leqslant N' \quad \text{or} \quad |u| = |Kz| \leqslant |K|N'$$

so that, taking $\alpha := |K|N'$, we have $|u| \leqslant \alpha$ for all sequences $v \in S(\delta^m)$. This implies that $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$. □

Proposition 10

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be a recursive system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where $f: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is differentiable at the origin and satisfies $f(0, 0) = 0$, and the initial condition is $x_0 = 0$. Let (F, G) be the jacobian matrix of the partial derivatives of f at the origin, where F is $m \times m$ and G is $m \times m$, and assume that the pair (F, G) is controllable. Then, the system Σ satisfies Condition 1.

Proof

Since the function f is differentiable at the origin, then

$$f(x, u) = Fx + Gu + \psi(x, u)$$

where

$$\lim_{|(x,u)| \to 0} \frac{\psi(x, u)}{|(x, u)|} = 0$$

Now, let γ be a non-singular $m \times m$ matrix, let $\delta > 0$ be a real number (whose value will be chosen later), and let v be an arbitrary sequence in $S(\delta^m)$. Let $u := \Sigma_\gamma^{-1} v$. Then, using the proof of Proposition 9, it follows that

$$\gamma u_{k+1} = (F - G\gamma^{-1})(\gamma u_k) + v_{k+1} - Fv_k - \psi$$

Now, let γ be the non-singular $m \times m$ matrix chosen in the proof of Proposition 9. Denoting $z := \gamma u$, $K := \gamma^{-1}$, and $w_k := v_{k+1} - Fv_k - \psi$, it follows, as in (3.6), that

$$z_{k+1} = (F - GK)z_k + w_k$$

which can be regarded as the state representation of a linear system with state z and input w, operating under the static state feedback K. Recalling that

$$\lim_{|(x,u)| \to 0} \frac{\psi(x, u)}{|(x, u)|} = 0$$

it follows that, for any real $\omega > 0$, there is a real number $\xi > 0$ for which

$$|\psi(x, u)| < \omega\xi$$

for all pairs (x, u) satisfying $|(x, u)| < \xi$. Now, assuming that $|(x, u)| < \xi$, then

$$|w| \leqslant |v| + |Fv| + |\psi| \leqslant \delta + |F|\delta + \omega\xi$$

Further, since (3.6) is a linear system, it follows that

$$z = r(z_0) + Z_0(w)$$

where $Z_0(w)$ is the response of the system (3.6) to the input sequence w with zero initial conditions, and $r(z_0)$ is the response to the zero input sequence from the initial condition z_0. Then, from the choice of K, the linear system (3.6) is strictly stable and hence there are bounded numbers $|r| > 0$ and $|Z_0| > 0$ such that

$$|Z_0(w)| \leqslant |Z_0|\,|w| \leqslant |Z_0|(\delta + |F|\delta + \omega\xi) \quad \text{and} \quad |r(z_0)| \leqslant |r|\,|z_0|$$

Next, using $v = x + \gamma u$, then $\gamma u = v - x$, and, since $x_0 = 0$, it follows that

$$|z_0| = |\gamma u_0| = |v_0| \leqslant \delta$$

so that

$$|r(z_0)| \leqslant |r|\delta \quad \text{and} \quad |z| \leqslant |r|\delta + |Z_0|(\delta + |F|\delta + \omega\xi)$$

Consequently,

$$\begin{aligned} |u| = |Kz| \leqslant |K|\,|z| &\leqslant |K|[|r|\delta + |Z_0|(\delta + |F|\delta + \omega\xi)] \\ &= (|K|\,|r| + |K|\,|Z_0| + |K|\,|Z_0|\,|F|)\delta + |K|\,|Z_0|\omega\xi \end{aligned}$$

Also,

$$\begin{aligned} |x| &\leqslant |v - \gamma u| \leqslant |v| + |\gamma u| = |v| + |z| \leqslant \delta + |r|\delta + |Z_0|(\delta + |F|\delta + \omega\xi) \\ &= (1 + |r| + |Z_0| + |Z_0|\,|F|)\delta + |Z_0|\omega\xi \end{aligned}$$

Now, choose $\xi > 0$ so that

$$\omega < \min\left\{\frac{1}{(2|K|\,|Z_0|)}, \frac{1}{(2|Z_0|)}\right\}$$

and choose $\delta > 0$ so that

$$\delta < \left(\frac{\xi}{2}\right) \min\left\{\frac{1}{(|K|\,|r| + |K|\,|Z_0| + |K|\,|Z_0|\,|F|)}, \frac{1}{(1 + |r| + |Z_0| + |Z_0|\,|F|)}\right\}$$

For the choice of δ, it is clear that

$$|u| < \xi \quad \text{and} \quad |x| < \xi$$

(so that also $|(x, u)| < \xi$) whenever $v \in S(\delta^m)$. However, $|u| < \xi$ implies that $\Sigma_\gamma^{-1}[S(\delta^m)] \subset S(\xi^m)$, or $S(\delta^m) \subset \Sigma_\gamma[S(\xi^m)]$ and Condition 1 holds with $\alpha := \xi$. □

Now consider the question of dynamics assignment, which is one of our major concerns in this paper. Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be a system satisfying Condition 1, and let γ be a matrix described in that condition. In order to be able to assign the dynamical behaviour of the closed-loop system, the following procedure is adopted, using the notation of Theorem 5. First, replace the unimodular transformation βI in

(3.4) by an arbitrary recursive, bicausal, unimodular, and uniformly l^∞-continuous transformation $M:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$ for which $M^{-1}[S((5\theta)^m)]\subset S(\zeta^m)$. (Recall that θ is a desired bound for the amplitudes of the input sequences of the final closed-loop system.) Now, the following systems are constructed:

$$\left.\begin{aligned} A &:= M-\varepsilon Q_*:S(\mathbb{R}^m)\to S(\mathbb{R}^m) \\ &\text{and} \\ B &:= \varepsilon I:S(\mathbb{R}^m)\to S(\mathbb{R}^m)\end{aligned}\right\} \tag{3.7}$$

Letting $S':=M^{-1}[S((5\theta)^m)]$, and recalling the discussion leading to Theorem 5, it follows that

$$APv+BQv=(M-\varepsilon Q)I_1v+\varepsilon Qv=(Mv-\varepsilon Qv)+\varepsilon Qv=Mv$$

for all elements $v\in S'$, and it is easy to see that A and B satisfy all the conditions of the Corollary. Thus, with the compensators $\pi:=B^{-1}=(1/\varepsilon)I$ and $\varphi:=A$, the configuration $\Sigma_{(\gamma,\pi,\varphi)}$ becomes internally stable for all input sequences bounded by θ.

Now, using the compensators π and φ constructed in the previous paragraph, consider the dynamical behaviour of the internally stable system $\Sigma_{\gamma(\pi,\varphi)}$, namely, the transmission from the input u to the output z in Fig. 2. Recalling that the coprime fraction representation $\Sigma_\gamma=PQ^{-1}$ used here has $P=I$ over the applicable subspace (due to the choice of γ), then from (1.5) it follows that the input/output relation induced by $\Sigma_{\gamma(\pi,\varphi)}$ satisfies

$$\Sigma_{\gamma(\pi,\varphi)}=M^{-1} \tag{3.8}$$

Thus, the dynamical behaviour of $\Sigma_{\gamma(\pi,\varphi)}$ can be assigned arbitrarily, subject only to the requirement that M is recursive, unimodular, bicausal, and uniformly l^∞-continuous. The explicit construction of the matrix γ is described in the proof of Propositions 4–10.

Regarding the internally stable closed-loop system $\Sigma_{(\gamma,\pi,\varphi)}$, namely, the transmission from the input u to the output y in Fig. 2, it follows from (2.7) that

$$\Sigma_{(\gamma,\pi,\varphi)}=P_\sigma M^{-1} \tag{3.9}$$

where P_σ is the (stable) numerator of the right coprime fraction representation $\Sigma=P_\sigma Q^{-1}$ of the original system Σ. Since $Q=\Sigma_\gamma^{-1}$ is used here then

$$P_\sigma=\Sigma Q=\Sigma\Sigma_\gamma^{-1}:S(\zeta^m)\to S(\mathbb{R}^m)$$

As seen from (3.9), the dynamical behaviour of $\Sigma_{(\gamma,\pi,\varphi)}$ is determined by the unimodular system M and by the (stable) numerator P_σ. The dynamics of P_σ depend on the choice of γ. Of course, interactions (or 'cancellations') between P_σ and M^{-1} are allowed here.

Remark

Note that, in view of the stability of M^{-1}, the condition $M^{-1}[S((5\theta)^m)]\subset S(\zeta^m)$ is simply an amplitude-scaling relation, and has no dynamical implications. To see why this is so, notice that the stability of M^{-1} implies that there is a real number $\xi>0$ such that $M^{-1}[S((5\theta)^m)]\subset S(\xi^m)$. In case $\xi>\zeta$, define the constant $\lambda:=\xi/\zeta$. Then, the system $M_\lambda:=M\lambda$, obtained by premultiplying the input sequences to M by λ, clearly satisfies

$$M_\lambda^{-1}[S((5\theta)^m)]\subset(1/\lambda)S(\xi^m)\subset S(\zeta^m)$$

Thus, by replacing M with M_λ, the amplitude restriction is satisfied without affecting the dynamical behaviour.

To summarize the discussion of the problem of dynamics assignment, it can be said that the dynamics of the internally stable system $\Sigma_{\gamma(\pi,\varphi)}$ can be arbitrarily assigned, and the following can be stated.

Theorem 6

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be a strictly causal homogeneous system, let Σ_γ be given by (3.1), and let $\theta > 0$ be a real number. Assume there is an $m \times m$ constant non-singular matrix γ and a real number $\delta > 0$ such that $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for some real $\alpha > 0$. Let $\zeta, \varepsilon > 0$ be real numbers satisfying $\zeta < \delta$ and $\varepsilon < \theta/\alpha$. Let $M: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be any unimodular, bicausal, and uniformly l^∞-continuous system satisfying $M^{-1}[S((5\theta)^m)] \subset S(\zeta^m)$. Then, for the compensators $\pi := B^{-1}$ and $\varphi := A$, where A and B are given by (3.7) and Q_* is given by (3.2), the closed-loop system $\Sigma_{\gamma(\pi,\varphi)}$ is internally stable for all input sequences bounded by θ, and it satisfies $\Sigma_{\gamma(\pi,\varphi)} = M^{-1}$. Furthermore, the system $\Sigma_{(\gamma,\pi,\varphi)}$ is also internally stable for all input sequences bounded by θ, and it satisfies $\Sigma_{(\gamma,\pi,\varphi)} = P_\sigma M^{-1}$.

Thus, it can be seen that roughly arbitrary assignment of dynamical properties is possible for non-linear systems. The compensators π and φ of Theorem 6 will be recursive whenever Σ and M are recursive, as seen from (3.7). A step-by-step description of the construction of π and φ is provided below. First, however, the class of solutions described by (3.7) is extended.

Employing the method described in (2.4), the class of solutions described by (3.7) can be widened as follows, again using the notation of Theorem 5. In view of (2.5) and its relevant discussion, a left fraction representation of the injective homogeneous system Σ_γ over the needed subspace is given by $\Sigma_\gamma = G^{-1}T$, where $G := \Sigma_\gamma^{-1}: \Sigma_\gamma[S(\alpha^m)] \to S(\alpha^m)$ and where $T := I: S(\alpha^m) \to S(\alpha^m)$ is the identity system. Now, let $h: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be any strictly causal, recursive, stable, and uniformly l^∞-continuous system, for which $|hv| < \theta$ for all $v \in S(\alpha^m)$. Recalling the definition of Q_* and (2.4), define the systems

$$\left.\begin{aligned} A' &:= M - (\varepsilon I + h)Q_*: S(\mathbb{R}^m) \to S(\mathbb{R}^m) \\ &\text{and} \\ B' &:= (\varepsilon I + h): S(\mathbb{R}^m) \to S(\mathbb{R}^m) \end{aligned}\right\} \tag{3.10}$$

Then, using the notation employed in the discussion of (2.4), and noticing that $Gv = Q_* v$ for all $v \in S(\zeta^m)$, it is easy to see that $A'Pv + B'Qv = Mv$ for all elements $v \in S'$ $(= M^{-1}[S((5\theta)^m)] \subset S(\zeta^m))$. In view of the strict causality of h and the fact that $\varepsilon \neq 0$, it is also easy to see that B' is bicausal and A' is causal, and that the conditions of the Corollary are satisfied. Thus, the compensators $\pi := B'^{-1}$ and $\varphi := A'$ yield internal stabilization of $\Sigma_{(\gamma,\pi,\varphi)}$ for all input sequences $u \in S(\theta^m)$. Moreover, we clearly still have $\Sigma_{\gamma(\pi,\varphi)} = M^{-1}$ and $\Sigma_{(\gamma,\pi,\varphi)} = P_\sigma M^{-1}$, as before.

A discussion follows on the general case of systems for which the dimension of the input space is not necessarily equal to the dimension of the output space. Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ be a strictly causal homogeneous system. The assumption at the beginning of this section, that the dimension of the input space of Σ is equal to the dimension of its output space, namely that $m = q$, can easily be eliminated as follows. First recall the general definition of the system Σ_γ given in (2.1), which is now rewritten for convenience.

$$\Sigma_\gamma := \gamma \mathscr{I}_1 + \mathscr{I}_2 \Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p), \quad p := \max\{m, q\} \tag{3.11}$$

Here, γ is a constant $p \times p$ invertible matrix. Recalling the definitions of $\mathscr{I}_1$ and $\mathscr{I}_2$ from § 2, the two cases are distinguished: $m \geqslant q$ and $m < q$. When $m \geqslant q$, then $p = \max\{m, q\} = m$, and the injection $\mathscr{I}_1$ is in fact the identity system$I : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$. This implies that $\operatorname{Im} \Sigma_\gamma = S(\mathbb{R}^p) = S(\mathbb{R}^m)$ and the previous discussion of the case $m = q$ also applies to this case.

Probably the easiest way to include the case where $q > m$ in the discussion is to add some 'dummy' inputs to the system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ so that the number of its inputs becomes equal to the number of its outputs. The easiest way to achieve this is as follows. Assume that $q > m$, and let $\Pi : \mathbb{R}^q \to \mathbb{R}^m$ be the standard projection onto the first m coordinates. With a slight abuse of notation, denote by Π the analogous projection $\Pi : S(\mathbb{R}^q) \to S(\mathbb{R}^m)$. Then, the system

$$\Sigma' := \Sigma\Pi : S(\mathbb{R}^q) \to S(\mathbb{R}^q) \tag{3.12}$$

has an input space of dimension equal to the dimension of its output space. It is very easy to implement the system Σ'. Indeed, given an input sequence $u \in S(\mathbb{R}^q)$ with $u = (u^1, u^2, \ldots, u^q)$, where $u^1, u^2, \ldots, u^q \in S(\mathbb{R})$, it follows that $\Sigma' u = \Sigma(u^1, u^2, \ldots, u^m)$, namely the last $q - m$ components of u are simply ignored. In practice, this simply amounts to not feeding these components into the system. Furthermore, it is easy to see that the system Σ' is recursive, strictly causal, and homogeneous whenever Σ is recursive, strictly causal, and homogeneous. Thus, all the discussion in this section applies to Σ', and stabilization with dynamics assignment for Σ' will evidently also yield stabilization with dynamics assignment for the original system Σ. The case where $q > m$ could also be dealt with in a more direct way without increasing the number of inputs, but this would involve a more lengthy discussion.

The procedure of stabilization and dynamics assignment is now summarized in the form of a step-by-step algorithm. For the sake of conciseness, the form of the compensators determined by the systems A and B and given by (3.7) is used. Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ be a recursive system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$ with the initial condition $x_0 = 0$.

Step 1

Choose a real number $\theta > 0$. This number will serve as the bound on the amplitude of the input sequences of the final stabilized closed-loop system.

Step 2

Let $p := \max\{m, q\}$, and notice that $p \geqslant m$. Let $\Pi : \mathbb{R}^p \to \mathbb{R}^m$ be the standard projection onto the first m co-ordinates, so that Π is the identity map when $p = m$. Define the system $\Sigma' := \mathscr{I}_2 \Sigma\Pi : S(\mathbb{R}^p) \to S(\mathbb{R}^p)$. Using the notational convention of the paragraph following (2.1), this system is represented by $x_{k+1} = f(x_k, \Pi u_k)$, $x_0 = 0$.

Step 3

Find a constant $p \times p$ non-singular matrix γ satisfying Condition 1 for Σ'. Some methods through which such a matrix can be computed are described in the proofs of

Propositions 4–10. Let $\delta > 0$ be a real number such that $S(\delta^p) \subset \Sigma'_\gamma[S(\alpha^p)]$ for some real $\alpha > 0$, as described in Condition 1. Notice that Σ'_γ is bicausal. Choose constant positive numbers $\zeta < \delta$ and $\varepsilon < \theta/\alpha$. Choose a recursive, unimodular, bicausal, and uniformly l^∞-continuous system $M: S(\mathbb{R}^p) \to S(\mathbb{R}^p)$ satisfying $M^{-1}[S((5\theta)^p)] \subset S(\zeta^p)$. The system M will determine the dynamical behaviour of the closed-loop system.

Step 4

Let $E: S(\mathbb{R}^p) \to S(\zeta^p)$ be the static system constructed in Lemma 2. Define the system $Q_* := \Sigma'^{-1}_\gamma E: S(\mathbb{R}^p) \to S(\alpha^p)$. This system is a combination of the two recursive systems Σ'^{-1}_γ (whose recursive representation is given by (2.2)) and E (whose recursive representation is given by (3.3)), and therefore can readily be implemented on a digital computer.

Step 5

Construct the system

$$A := M - \varepsilon Q_*: S(\mathbb{R}^p) \to S(\mathbb{R}^p)$$

and

$$B := \varepsilon I: S(\mathbb{R}^p) \to S(\mathbb{R}^p)$$

where $I: S(\mathbb{R}^p) \to S(\mathbb{R}^p)$ is the identity system, as in (3.7). Using these systems, construct the precompensator $\pi = B^{-1} = (1/\varepsilon)I$, which is simply an amplifier in this case, and the feedback compensator $\varphi := A = M - \varepsilon Q_*$. Then, the closed-loop system $\Sigma'_{(\gamma,\pi,\varphi)}$ around Σ' will be internally stable for all input sequences from $S(\theta^p)$, and $\Sigma'_{\gamma(\pi,\varphi)} = M^{-1}$ and $\Sigma'_{(\gamma,\pi,\varphi)} = P_\sigma M^{-1}$.

Of course, similar steps can also be used in conjunction with (3.10) to obtain more general compensators.

The discussion is concluded with an example on the computation of the stabilizing compensators π and φ. As stressed throughout this paper, and as will be clearly seen in the example, this procedure yields the stabilizing compensators π and φ in an explicit form and in terms of quantities that are directly derived from the given recursion functions of the original system Σ and the unimodular system M. In order to avoid cluttering the presentation unnecessarily, a rather simple example is provided. The situation in general is similar.

Example

Consider the system $\Sigma: S(\mathbb{R}) \to S(\mathbb{R})$ described by the recursive representation

$$x_{k+1} = 2x_k + \cos(x_k u_k) + u_k, \quad x_0 = 0$$

It is easy to see that this system is not stable. Follow Steps 1–5 of the stabilization procedure to compute the stabilizing compensators. First, choose $\theta = 1$. The recursion function here is

$$f(x, u) = 2x + \cos(xu) + u$$

so that the conditions of Proposition 5 are satisfied with $\psi = \cos(xu)$. Thus, δ can be chosen as any real positive number, so, for example, take $\delta = 2$. Further, following the notation of the proof of Proposition 5, $a = 2$ and $b = 1$ so that

$$n(\gamma) = \left|\frac{1}{\gamma} - 2\right|$$

and, choosing $\gamma = \frac{1}{2}$, then $n(\gamma) = 0$, so the requirement $0 \leqslant |n(\gamma)| < 1$ is satisfied. For this choice, it follows from the proof of Proposition 5 that $c = 2\delta + 1 = 5$. Using (3.5) and the fact that $n(\gamma) = 0$, we have

$$|\gamma u_{k+1}| \leqslant |n(\gamma)|\,|\gamma u_k| + (\delta + c) = 7$$

Since $\gamma = \frac{1}{2}$ then $|u| \leqslant 14$, or $S(2) \subset \Sigma_\gamma[S(14)]$, and $\alpha = 14$. Now, choose $\zeta = 1$, and $M = \beta I$ as in (3.4) and Theorem 5. Since $\beta \geqslant 5$ must be true, choose $\beta = 5$. Also, since $\varepsilon < \frac{1}{14}$ must also be true, choose $\varepsilon = \frac{1}{20}$. Next, the system Q_* is given by the following equations, where w is the input sequence of Q_* and $v := Q_* w$ is the output sequence. Here, since $\zeta = 1$, the function e is defined by $e(w_k) := w_k$ if $|w_k| \leqslant 1$ and $e(w_k) := \operatorname{sign}(w_k)$ if $|w_k| > 1$, where $\operatorname{sign}(\cdot) = \pm 1$, depending on the sign of the argument.

$$z_k = e(w_k), \quad k = 0, 1, 2, \ldots$$

$$v_{k+1} = 2\{z_{k+1} - 2z_k - \cos[(z_k - v_k/2)v_k]\}, \quad k = 0, 1, 2, \ldots$$

$$v_0 = 2[z_0 - x_0] = 2z_0$$

Finally, with β, ε and Q_* available, the systems A and B of (3.4) can directly be constructed, and from them the compensators $\pi = B^{-1}$ and $\varphi = A$. Note that the precompensator π here is simply an amplifier of 20. Regarding φ, for any input sequence w, the output sequence $s := \varphi w$ it generates is given by $s_k = 5w_k - (\frac{1}{20})v_k$, $k = 0, 1, 2, \ldots$, where the sequence $v := Q_* w$ is given by the above recursive equations. The closed-loop system $\Sigma_{(\gamma,\pi,\varphi)}$ here is internally stable for all input sequences bounded by 1 (since $\theta = 1$ was chosen), and, since $M = \beta$, it easily follows that

$$\Sigma_{\gamma(\pi,\varphi)} = \frac{1}{\beta} = \frac{1}{5}$$

which is a static system. This gives a demonstration of the wide range of dynamics assignment that is obtainable. Notice however that $\Sigma_{(\gamma,\pi,\varphi)}$ here is not static.

Finally, the following remark is made. Throughout this discussion, the quantity γ in the control configuration in Fig. 2 has been interpreted as an invertible constant matrix. In some cases, better results can be obtained by using a continuous and continuously invertible non-linear function $\gamma(u)$ in place of γ, or possibly even by using an appropriate dynamical system.

Acknowledgments

This research was supported in part by the National Science Foundation, Grant Numbers 8501536 and 8896182. Starting August 1987, this research was also supported in part by U.S. Air Force Grant AFOSR-85-0186, by Office of Naval Research Grant N00014-86-K-0538, and by U.S. Army Grant DAAG29-85-K-0099 through the Center for Mathematical System Theory, University of Florida, Gainesville, FL 32611, U.S.A.

REFERENCES

DESOER, C. A., and CHAN, W. S., 1975, *J. Franklin Inst.*, **300,** 335.
DESOER, C. A., and LIN, C. A., 1984, *Int. J. Control*, **40,** 37.
HAMMER, J., 1983 a, *Int. J. Control*, **37,** 37; 1983 b, *Ibid.*, **37,** 63; 1984 a, *Ibid.*, **40,** 1; 1984 b, *Ibid.*, **40,** 953; 1985 a, *Ibid.*, **42,** 1; 1985 b, The concept of rationality and stabilization of nonlinear systems. *Proc. Symp on Mathematical Theory of Networks and Systems*, Stockholm, Sweden, edited by C. Byrnes and A. Lindquist (Amsterdam: North-Holland); 1986, *Int. J. Control*, **44,** 1349; 1987, *Ibid.*, **46,** 455.
KURATOWSKI, K., 1961, *Introduction to Set Theory and Topology* (New York: Pergamon).
ROSENBROCK, H. H., 1970, *State Space and Multivariable Theory* (London: Nelson).
SONTAG, E. D., 1981, *Inf. Control*, **51,** 105.
VIDAYSAGAR, M., 1980, *I.E.E.E. Trans. autom. Control*, **25,** 504.
WONHAM, W. M., 1967, *I.E.E.E. Trans. autom. Control*, **12,** 660.