INT. J. CONTROL, 2000, VOL. 73, NO. 18, 1671–1685

Approximate model matching for non-linear control systems

JACOB HAMMER†

Approximate model matching refers to the problem of controlling a non-linear system so as to achieve a response resembling that of a desirable model. The paper presents a family of recursive output feedback controllers that can achieve approximate model matching in all cases where it is possible. The design of these controllers depends on the solution of a set of algebraic inequalities.

1. Introduction

A perennial question facing the non-linear control engineer is how to specify design objectives for a non-linear control system. As is well known, characteristics of non-linear systems can be quite complex; they do not usually lend themselves to the simple characterizations possible for linear systems, like pole locations, phase margin, or gain margin. Perhaps one of the most effective ways of specifying the desired behaviour of a non-linear system is by requiring the system to resemble a specified model. An appropriate model can be determined through computer simulation, or by qualitative considerations. Our ojective is to design a controller to control a given plant so that the plant–controller combination approximates the specified model. We refer to this objective as 'approximate model matching'.

More concretely, consider a non-linear plant Σ that needs to be controlled so as to perform a specific task. Suppose that as a result of computer simulation or other considerations, one has determined a model M whose characteristics are suitable for performing the required task. Our objective is to design a controller C which, when combined with the plant Σ, creates a controlled system Σ_c that 'resembles' the model M. The 'resemblance' must be robust in the sense that it is preserved under disturbances and parameter uncertainties. Of course, an accurate meaning must be given to the term 'resembles', as we now proceed to do.

First, let $\Im$ be the class of input signals for which the controlled system Σ_c needs to 'resemble' the model M. Normally, the set $\Im$ consists of all input signals of amplitude not exceeding a specified bound $\theta > 0$.

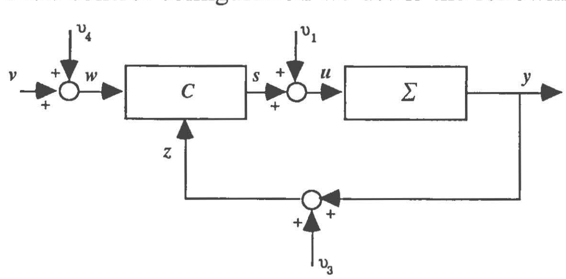
Now, for an input signal $v \in \Im$, the response of the controlled system is $\Sigma_c v$ and the response of the model is Mv. Our objective is to design the controller C so that $\Sigma_c v$ is 'close' to Mv for all input sequences $v \in \Im$. Explicitly, let $|u|$ denote the amplitude of a signal u, and let $\Delta > 0$ be a real number. We say that Σ_c is a Δ-*approximant* of M if

$$|\Sigma_c v - Mv| \leq \Delta \tag{1.0.1}$$

for every input signal $v \in \Im$. In other words, Σ_c is a Δ-approximant of M if the amplitude of the discrepancy among the responses of Σ_c and of M does not exceed Δ for any input sequence of interest. Clearly, a smaller value of Δ yields a better approximant of the model M. The desired model M and the value of the maximal discrepancy Δ are specified as design objectives.

The paper addresses several facets of the approximate model matching problem. These include necessary and sufficient conditions for the existence of a controller C that achieves approximate model matching (§ 3); the characterization of a family of simple controllers from which C can be selected (§§ 1 and 3); and the development of computational techniques for the design of approximate model matching controllers (§ 4). The characterization of a family of simple controllers is of particular interest, as it reduces the realm of controllers that need to be considered as candidates in design practice. This simplifies the design task, and facilitates the use of numerical experimentation as a design tool in non-linear control.

Our discussion concentrates on the case where the plant Σ that needs to be controlled is a discrete-time recursive system. The results also apply to continuous time systems within a sampled-data framework. The basic control configuration we use is the following.



(1.0.2)

Here, Σ is the plant that needs to be controlled; C is a causal feedback controller that generates the input sequence of Σ based on the output sequence y and the

Received 1 August 1998. Revised 1 August 1999. Communicated by Professor V. Kucera.

† Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611-6130, USA.

external input sequence v of the composite system. The signals v_1, v_3 and v_4 represent disturbance signals. The only *a priori* information available about each disturbance signal v_i is that its amplitude does not exceed a specified bound $n_i > 0$. The disturbance amplitude bounds can be large, and are not considered infinitesimal. The closed loop system Σ_c is required to be a Δ-approximant of M, despite the presence of the disturbance signals v_1, v_3 and v_4.

Clearly, the question of whether or not there is a controller C that turns Σ_c into a Δ-approximant of M depends, among other factors, on the values of the disturbance amplitude bounds $\{n_i\}$. It is possible, of course, that the disturbance amplitudes are so large that no controller can maintain a discrepancy not exceeding Δ between Σ_c and M with such disturbances. In some applications, it is of interest to find the largest disturbance amplitude bounds $\{n_i\}$ for which there exists a controller C that turns Σ_c into a Δ-approximant of M. This issue is discussed in § 4.

The plants Σ we consider are strictly causal recursive non-linear systems. It will be convenient to separate this class of systems into two subclasses: systems whose state is provided as output, and systems whose output is not a state. To start with the first case, consider a system Σ given in terms of a nominal *state representation* of the form

$$y_{k+1} = f(y_k, u_k), \qquad k = 0, 1, 2, \ldots \tag{1.0.3}$$

Here, f is a function, called the *recursion function* of Σ. The initial condition of Σ is y_0. A system that has a representation of the form (1.0.3) is called an *input/state* system. The model M that needs to be approximately matched in this case is also given in terms of a nominal state representation of the form

$$\xi_{k+1} = \varphi(\xi_k, v_k), \qquad k = 0, 1, 2, \ldots \tag{1.0.4}$$

We assume that ξ_k is a vector of the same dimension as y_k. The initial condition of the model M is ξ_0, and is not necessarily identical to the initial condition y_0 of the system Σ.

Now, let y_k be the output value of the composite system Σ_c at the time-step k in response to the input sequence v. Condition (1.0.1) then becomes the requirement

$$|y_k - \xi_k| \leq \Delta$$

for all integers $k \geq 0$, for all input sequences $v \in \mathfrak{I}$, for all permissible disturbance signals, and for all permissible initial conditions.

In realistic design situations, one cannot assume that the recursion function f of Σ is known with absolute accuracy. To take this fact into account, we incorporate an additive disturbance v_2 into the recursive representation of Σ as

$$y_{k+1} = f(y_k, u_k) + v_{2,k}, \qquad k = 0, 1, 2, \ldots \tag{1.0.5}$$

The amplitude of the disturbance v_2 does not exceed a given bound $n_2 > 0$. The controller C is required to make the closed loop system Σ_c into a Δ-approximant of the model M for all disturbances v_1, v_2, v_3 and v_4, as long as these disturbances do not exceed their respective amplitude bounds.

In § 3 we provide necessary and sufficient conditions for the existence of a controller C that fulfills this design objective. When such a controller exists, we provide a technique for its design in § 4. From a computational standpoint, the determination of the existence of C, as well as the calculation of C, depend upon the solution of a set of algebraic inequalities derived directly from the given recursion functions f and φ. The computational techniques employed here are extensions of the techniques developed in Hammer (1998).

From a numerical standpoint, the use of computational techniques that are based on the solution of inequalities is usually more efficient than the use of techniques based on the solution of differential equations (like classicial optimization techniques). The numerical solution of differential equations requires repeated iteration, which leads to large numerical errors when the discretization interval is not small enough. In many cases, the discretization interval used for the numerical solution of inequalities can be larger, since the solution does not require iteration. A larger discretization interval lowers the computational burden.

An important qualitative issue that can be addressed in this context is the following. Is there a simple family of controllers whose members can achieve approximate model matching in all cases where approximate model matching is possible. In § 3 we show that such a controller family does indeed exist. Using the notation of (1.0.2), the members of this controller family are of the form

$$C: \quad \begin{cases} \zeta_{k+1} = \varphi(\zeta_k, w_k) + v_{5,k} \\ s_k = \sigma(z_k, \zeta_k, w_k), \qquad k = 0, 1, 2, \ldots \end{cases} \tag{1.0.6}$$

Here, φ is the recursion function of the model M, and σ is a feedback function that is derived as part of the design of the controller. The term v_5 represents a disturbance signal that originates from inaccuracies in the implementation of the function φ within the controller. As was the case with the other disturbances, the only information available about v_5 is that its amplitude does not exceed a given bound $n_5 \geq 0$.

The controller C of (1.0.6) is a dynamic feedback controller. As we can see, it contains an implementation of the model M that needs to be approximately matched, as well as a feedback function σ. The computation of σ is discussed in § 4 below. In § 3 we show that, whenever the approximate model matching problem for

Σ and M is solvable, it can be solved by using a controller of the form (1.0.6). Thus, (1.0.6) constitutes a complete family of controllers for the solution of the approximate model matching problem with state output. This observation is of practical importance, as it vastly reduces the number of controller candidates that need to be examined in practice.

The feedback function σ of (1.0.6) depends on the recursion functions f and φ, as well as on the various disturbance bounds and the permissible error bound Δ. The main step in the calculation of σ is the solution of a set of simultaneous algebraic inequalities, derived from the given recursion functions f and φ (see § 4 below). The computational techniques developed in the present paper rely upon a generalization of the concept of an eigenset, introduced in Hammer (1989, 1998).

The disturbance signals υ_1, υ_2, υ_3, υ_4 and υ_5 may originate from a variety of sources, and each disturbance signal may be the culmination of the effects of a number of disturbance sources. These disturbance sources may include physical noise sources as well as quantization errors incurred when the controller C is implemented on a digital computer. For a more detailed discussion of various possible origins of disturbance signals, see Hammer (1998, § 1).

We turn our attention now to the more general case of systems whose output value is not a state.

1.1. *Recursive systems and formal realizations*

Consider the approximate model matching problem for the case where the system Σ is described by a nominal recursive representation of the form

$$y_{k+\eta+1} = f(y_{k+\eta}, y_{k+\eta-1}, \ldots, y_k, u_{k+\mu}, u_{k+\mu-1}, \ldots u_k),\quad k = 0, 1, 2, \ldots \tag{1.1.1}$$

Here, f is a function called the *recursion function* of Σ, and η and μ are two non-negative integers satisfying $\mu \le \eta$. To distinguish from (1.0.3), we assume $\eta \geq 1$. The *recursion order* of (1.1.1) is $\eta + 1$. At the step k, the output value is y_k and the input value is u_k. The initial conditions of Σ are $y_o, \ldots y_{k+\eta}$; it is not assumed that the initial conditions are known accurately. The condition $\mu \le \eta$ implies that the system Σ is strictly causal.

The model M to be approximately matched is also given in terms of a nominal recursive representation

$$\xi_{k+\eta+1} = \varphi(\xi_{k+\eta}, \xi_{k+\eta-1}, \ldots, \xi_k, v_{k+\mu}, v_{k+\mu-1}, \ldots, v_k),\quad k = 0, 1, 2, \ldots, \tag{1.1.2}$$

where φ is the recursion function of M. At the step k, the output value of M is ξ_k and its input value is v_k, where ξ_k is of the same dimension as the output value y_k for Σ. For the moment, we assume that M and Σ share the same recursion order $\eta + 1$, but this assumption is released in § 2 below.

To take into account disturbances and inaccuracies within the recursive representation of the system Σ, we incorporate a disturbance signal υ_2

$$y_{k+\eta+1} = f(y_{k+\eta}, y_{k+\eta-1}, \ldots, y_k, u_{k+\mu}, u_{k+\mu-1}, \ldots, u_k) + \upsilon_{2,k+\eta},\quad k = 0, 1, 2, \ldots \tag{1.1.3}$$

As before, the only *a priori* information available about the disturbance υ_2 is that its amplitude does not exceed a specified bound $n_2 \geq 0$.

The present more general situation can be readily reduced to the case of systems with state output. This is accomplished through the notion of a 'formal realization', as follows. Consider a system Σ having a nominal recursive representation of the form (1.1.1.). Define the *formal state* x_k of Σ at the step k by the vector

$$x_k := (y_k, y_{k-1}, \ldots, y_{k-\eta}, u_{k-\eta+\mu-1}, u_{k-\eta})^{\mathrm{T}} \tag{1.1.4}$$

where $^{\mathrm{T}}$ indicates the transpose. When the step k is regarded as the 'present', it follows from the inequality $\mu \le \eta$ that the present value x_k of the formal state is determined by present and past output values $y_k, y_{k-1}, \ldots, y_{k-\eta}$ of Σ, and by past input values $u_{k-\eta+\mu-1}, \ldots, u_{k-\eta}$ of Σ. Note that the formal state x_k contains all the input and output data necessary for the computation of the next step y_{k+1} of Σ, except for the latest relevant input value $u_{k+\mu-\eta}$.

Combining the formal state with (1.1.1), we obtain the recursive representation

$$x_{k+1} = F(x_k, \lambda_k),\qquad k = 0, 1, 2, \ldots \tag{1.1.5}$$

Here $\lambda_k := u_{k+\mu-\eta}$, and the function F is given in terms of the recursion function f and the components $y_k, y_{k-1}, \ldots, y_{k-\eta}, u_{k-\eta+\mu-1}, \ldots, u_{k-\eta}$ of x_k by (1.1.6) below. Note that the requirement $\mu \le \eta$ implies that the sequence λ is either equal to the input sequence u, or is a delay of u by $(\eta - \mu)$ steps, so that everything is causal.

$$x_{k+1} = \begin{pmatrix} y_{k+1} \\ y_k \\ \cdots \\ y_{k-\eta+1} \\ u_{k-\eta+\mu} \\ u_{k-\eta+\mu-1} \\ \cdots \\ u_{k-\eta+1} \end{pmatrix}$$

$$
= \begin{pmatrix} f(y_k, y_{k-1}, \dots, y_{k-\eta}, \lambda_k, u_{k-\eta+\mu-1}, \dots, u_{k-\eta}) \\ y_k \\ \dots \\ y_{k-\eta+1} \\ \lambda_k \\ u_{k-\eta+\mu-1} \\ \dots \\ u_{k-\eta+1} \end{pmatrix} \tag{1.1.6}
$$

We refer to (1.1.5) as a *formal realization* of Σ. A formal realization is a realization of the system in the usual sense, although it may not be a minimal realization.

The notion of formal realization is of important practical significance in the theory of nonlinear recursive systems. It provides a simple and useful mechanism for deriving a realization under very general conditions. The deficiency of not being a minimal realization is of secondary consequence here, since finding a minimal realization for a non-linear system is a complex and largely unknown process. The formal state also has the important advantage of being a combination of present and past output and input values of the system. In this way, feedback involving the formal state can be implemented directly by using available input and output values, without the need for an observer. This is significant for non-linear systems, where observer theory is quite incomplete.

Formal realizations are useful for the solution of the approximate model matching problem for non-linear recursive systems. Indeed, construct a formal realization of the system Σ of (1.1.1) and of the model M of (1.1.2). Then, apply to these realizations the controller of (1.0.6). Next, using the definition (1.1.4) of the formal state, substitute into the controller formula (1.0.6) the formal states in terms of the actual output values and input values of the systems at hand. Finally, noting that $u = s + v_1$ by diagram (1.0.2), we obtain a controller C of the form (1.1.7) (see below) (including the effects of the disturbances). As we can see, the formal realization provides us with simple means to generalize results from the theory of state feedback to the more general case of input/output control of non-linear recursive systems. The calculation of the function σ is discussed in §§3 and 4 below. This calculation is based on the solution of a set of algebraic inequalities derived from the given recursion functions f and φ.

We can distinguish among two constituents of the controller C of (1.1.7): one, represented by the first row of (1.1.7), is simply a simulation of the model M that needs to be approximately matched; the other, represented by the second row of (1.1.7), consists of a dynamic output feedback controller induced by the function σ. Note that the inequality $\mu \le \eta$ implies that C is strictly causal. The fact that (1.0.6) and (1.1.7) contain a model of M is a manifestation of the internal model principle (Wonham 1974).

We observe that (1.0.6) and (1.1.7) are, in fact, templates of universal families of controllers that can be used to achieve approximate model matching for a rather general class of non-linear systems. Such templates are valuable in practice, where they can be used as a basis for numerical experimentation with design parameters.

The most common implementation of a non-linear controller is in the form of a discrete system programmed on a digital computer. For such an implementation, the feedback function σ of (1.0.6) or (1.1.7) is only used over a discrete grid. As a consequence, continuity properties of σ are not of critical importance, and we shall not dwell on them in our present discussion.

The material of the present paper is a continuation of the work presented in Hammer (1989b, 1998) on the use of eigensets in the global theory of the robust control of non-linear systems. Alternative treatments of the global control of non-linear systems are described in Hammer (1984a, b, 1985, 1989a, 1991, 1994), Desoer and Kabuli (1988), Verma (1988), Sontag (1989), Chen and de Figueiredo (1990), Paice and Moore (1990), Verma and Hunt (1993), Paice and van der Schaft (1994), Baramov and Kimura (1995), the references cited in these papers, and others.

The paper is organized as follows. Section 2 introduces the basic notation and setup; provides a formal statement of the approximate model matching problem; and discusses the issue of differing recursion orders. The underlying concept of our discussion, namely, the concept of relative eigensets, is introduced and examined in §3. Section 4 develops computational tools for the design of controllers that achieve approximate model matching. The paper concludes with §5 that contains proofs of some results of earlier sections.

2. Basics

In this section we review our notation, and provide a formal statement of the design problem addressed in this paper. Let R^m be the set of all m-dimensional real

$$
C: \begin{cases} \zeta_{k+\eta+1} = \varphi(\zeta_{k+\eta}, \zeta_{k+\eta-1}, \dots, \zeta_k, w_{k+\mu}, w_{k+\mu-1}, \dots, w_k) + v_{5,k} \\ s_k = \sigma(z_k, \dots, z_{k-\eta}, (s+v_1)_{k-\eta+\mu-1}, \dots, (s+v_1)_{k-\eta}, \zeta_k, \dots, \zeta_{k-\eta}, w_{k-\eta+\mu-1}, \dots, w_{k-\eta}) \end{cases} \tag{1.1.7}
$$

vectors, and let $S(R^m)$ be the set of all sequences $u_0, u_1, u_2, \ldots$ of real vectors $u_i \in R^m, i = 0, 1, 2, \ldots$. A discrete-time system Σ that accepts input sequences of m dimensional real vectors and generates output sequences of p dimensional real vectors induces a map $\Sigma : S(R^m) \to S(R^p)$.

We shall use the l^∞-norm to characterize disturbances and their effects. To introduce the notation, consider a vector $v = (v^1, v^2, \ldots, v^m) \in R^m$; denote by

$$|v| := \max\{|v^1|, |v^2|, \ldots, |v^m|\}$$

the maximal absolute value of a component. Then, for a sequence $u \in S(R^m)$

$$|u| := \sup_{i \geq 0} |u_i|$$

is the l^∞-norm of u. It will be convenient to denote by $S(\theta^m)$ the set of all sequences $u \in S(R^m)$ satisfying $|u| \leq \theta$, where $\theta > 0$ is a real number; then $S(\theta^m)$ is the set of all sequences of amplitude not exceeding θ.

2.1. *Statement of the problem*

We turn now to a formal examination of the control configuration (1.0.2). Here, Σ is the system that needs to be controlled. We assume that Σ is a recursive system, described by a representation of the form (1.0.5) or (1.1.3). The response of the system Σ depends on the initial conditions and on the disturbance signal υ_2, as well as on the input sequence u. It will be convenient to denote by c_0 the initial condition of Σ, so that $c_0 = y_0$ when Σ is represented by (1.0.5), and $c_0 = (y_0, \ldots, y_\eta)$ when Σ is represented by (1.1.3). Then, denote by $\Sigma(c_0, \upsilon_2) : S(R^m) \to S(R^p) : u \mapsto \Sigma(c_0, \upsilon_2, u) = y$ the input/output map induced by Σ when the initial condition is c_0 and the disturbance signal is υ_2.

It is not assumed here that the initial condition c_0 of Σ is known with accuracy. In fact, one of the topics addressed in the ensuing discussion is the effect uncertainties about the initial condition on the performance of the closed loop system. The controllers designed in the sequel are able to accommodate uncertainties in the values of the initial condition of Σ, as long as the amplitude of these uncertainties does not exceed a specified bound.

The model M that needs to be approximately matched is described by a recursive representation of the form (1.0.4) or (1.1.2), and its response also depends, of course, on initial conditions. It will be convenient to denote by ϖ_0 the initial conditions of M; when M is described by (1.0.4) we have $\varpi_0 = \xi_0$, whereas when M is described by (1.1.2) we have $\varpi_0 = (\xi_0, \ldots, \xi_\eta)$. The response of the disturbed copy of M within the controllers (1.0.6) or (1.1.7) also depends on the disturbance signal υ_5. Let $M(\varpi_0, \upsilon_5) : S(R^m) \to S(R^p) : v \mapsto M(\varpi_0, \upsilon_5, v)$ be the input/output map induced by this copy of M when the initial conditions are ϖ_0 and the disturbance signal is υ_5. We do not assume that the initial conditions ϖ_0 of M are known precisely.

As mentioned in the previous section, the controller C we use to achieve our control objective is of the form (1.0.6) or (1.1.7). As can be seen, C is a dynamical system that has initial conditions, and its response also depends on the disturbance signal υ_5. Denoting by χ_0 the initial conditions of C, the controller induces a map $C(\chi_0, \upsilon_5) : S(R^m) \times S(R^p) \to S(R^m) : (w, z) \mapsto C(\chi_0, \upsilon_5, w, z) = s$. Uncertainties about the values of the initial conditions of C are permitted.

The response of the closed loop system (1.0.2) depends on all the quantities c_0, χ_0, v, υ_1, υ_2, υ_3, υ_4, and υ_5, and we shall denote it by $\Sigma_c(c_0, \chi_0, v, \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$. The output sequence y of (1.0.2) is then given by

$$y = \Sigma_c(c_0, \chi_0, v, \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$$

From a practical standpoint, our interest is restricted to the case where all initial conditions and all signals are bounded. In particular, we require all initial conditions to be bounded by a specified real number $r > 0$, so that $|c_0| \leq r, |\varpi_0| \leq r$, and $|\chi_0| \leq r$. Also, the amplitude of the external input sequence v is bounded by a specified real number $\theta > 0$, so that only input signals satisfying $|v| \leq \theta$ are permitted.

In the same spirit, we assume that the model M that needs to be approximately matched is BIBO (Bounded-Input Bounded-Output)-stable over its entire range of permissible initial conditions and input sequences. Specifically, there is a real number $\Theta > 0$ such that

$$|M(\varpi_0, \upsilon_5, v)| \leq \Theta \tag{2.1.1}$$

for all initial conditions satisfying $|\varpi_0| \leq r$, for all disturbances $|\upsilon_5| \leq n_5$, and for all input sequences satisfying $|v| \leq \theta + n_4$, where n_4 and n_5 are the amplitude bounds of the disturbances υ_4 and υ_5, respectively. We then say that M is *output bounded* by Θ. To simplify notation, it will be convenient to assume that

$$r \leq \Theta$$

In our discussion, we shall require the closed loop system Σ_c to approximate the response of the disturbed model $M(\varpi_0, \upsilon_5)$. Explicitly, we require

$$|\Sigma_c(c_0, \chi_0, v, \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5) - M(\varpi_0, v_5, v + \upsilon_4)| \leq \Delta \tag{2.1.2}$$

for all permissible initial conditions, disturbance signals, and input signals. This tacitly assumes that M has been chosen with satisfactory disturbance response properties, as the closed loop system Σ_c is required to approximate the response of M to disturbances as well.

From diagram (1.0.2) it follows that the closed loop system is described by the equations

$$\left.\begin{aligned} y &= \Sigma(c_0, v_2, s + v_1), \\ s &= C(\chi_0, v_5, y + v_3, v + v_4) \end{aligned}\right\} \qquad (2.1.3)$$

where all quantities affecting the response are made explicit. It is well known that the equations (2.1.3) have a unique solution whenever at least one of the systems Σ or C is strictly causal (e.g. Hammer 1994). In our discussion, the system Σ is described by (1.0.3) or by (1.1.1), so it is always strictly causal, and the closed loop system is well defined. In formal terms, we can now state our design objective as follows.

(2.1.4) **Approximate Model Matching Problem:** Let $\Sigma, M : S(R^m) \to S(R^p)$ be two systems, and let θ, Δ, r, ρ, n_0 and n_1 be positive real numbers. Find a controller C for which

$$|\Sigma_c(c_0, \chi_0, v, v_1, v_2, v_3, v_4, v_5) - M(\varpi_0, v_5, v)| \leq \Delta \qquad (2.1.5)$$

for all initial conditions, input signals, and disturbance signals satisfying:

(i) $|c_0| \leq r$, $|\varpi_0| \leq r$ and $|\chi_0| \leq r$;
(ii) $|\varpi_0 - \chi_0| \leq \rho$;
(iii) $|v| \leq \theta$;
(iv) $|v_1| \leq n_0, |v_2| \leq n_1, |v_3| \leq n_1, |v_4| \leq n_0, |v_5| \leq n_0$.

Find the largest disturbance amplitudes n_0, n_1 for which such a controller C exists, given the values of Δ and r.

Note that the requirement (ii) of (2.1.4) indicates that a discrepancy of magnitude not exceeding ρ is permitted between the initial conditions of M and the initial conditions of its replica within the controller C. This discrepancy may originate from inaccurate data about the initial conditions of M, or from inaccuracies in the process of setting the initial conditions of the controller. Also, condition (iv) distinguishes between two disturbance amplitude bounds: n_0 and n_1. We shall see later that the various disturbances naturally fall into two classes, each one with its own amplitude bound, as condition (iv) stipulates.

In some applications, the disturbance amplitude bounds n_0 and n_1 are not specified. Instead, one is required to determine the largest values of these bounds for which there exists a controller C satisfying the design objective (2.1.5), with specified values of Δ, ρ, and θ. The calculation of the maximal permissible disturbance amplitudes is discussed in § 4.

2.2. *Matching systems with different recursion orders*

Consider the approximate model matching problem for the case where the systems Σ and M have different recursion orders. Specifically, assume $\Sigma : S(R^m) \to S(R^p)$ is described by the nominal representation

$$y_{k+\alpha+1} = f(y_{k+\alpha}, y_{k+\alpha-1}, \ldots, y_k, u_{k+\beta}, u_{k+\beta-1}, \ldots, u_k), \quad k = 0, 1, 2, \ldots$$

whereas the model $M : S(R^m) \to (S(R^p)$ is given by

$$\xi_{k+\gamma+1} = \varphi'(\xi_{k+\gamma}, \xi_{k+\gamma-1}, \ldots, \xi_k, v_{k+\omega}, v_{k+\omega-1}, \ldots, v_k), \quad k = 0, 1, 2, \ldots$$

Here, the recursion order of Σ is $\alpha + 1$ and the recursion order of M is $\gamma + 1$. As mentioned earlier, we require both Σ and M to be strictly causal systems, so that

$$0 \leq \beta \leq \alpha, \qquad 0 \leq \omega \leq \gamma$$

One can formally equalize the recursion orders of Σ and M by inducing a shift of $|\alpha - \gamma|$ steps in the system having the lower recursion order. Explicitly, define two integers η, $\mu \geq 0$ by setting

$$\eta := \max\{\alpha, \gamma\},$$

$$\mu := \begin{cases} \max\{\beta, \omega + (\alpha - \gamma)\} & \text{if } \alpha \geq \gamma \\ \max\{\beta + (\gamma - \alpha), \omega\} & \text{if } \alpha < \gamma \end{cases}$$

Then, when $\alpha \geq \gamma$, we have $\eta = \alpha$, and, by inducing a shift of $(\alpha - \gamma)$ steps in M, we obtain the recursive representation.

$$\xi_{k+\eta+1} = \varphi'(\xi_{k+\eta}, \xi_{k+\eta-1}, \ldots, \xi_{k+\alpha-\gamma}, v_{k+\mu}, v_{k+\mu-1}, \ldots, v_{k+\alpha-\gamma})$$

which we formally rewrite in the form

$$\xi_{k+\eta+1} = \varphi(\xi_{k+\eta}, \xi_{k+\eta-1}, \ldots, \xi_k, v_{k+\mu}, v_{k+\mu-1}, \ldots, v_k)$$

This yields for M a recursive representation of the same recursion order as that of Σ.

For the case where $\alpha < \gamma$, we perform a shift by $(\gamma - \alpha)$ delay steps on the recursive representation of Σ, to obtain again a situation where Σ and M are both given by recursive representations of the same recursion order. Thus, we can always formally represent M and Σ by recursive representations of the same recursion order.

3. Approximate model matching

The present section deals with the approximate model matching problem for the case where the systems Σ and M are given in terms of state representations. The results can be applied to the more general case of recursive systems by using the notion of formal realization discussed in § 1.1. We start by introducing the notion of 'relative eigenset', which is the basic concept on which our framework is based.

3.1. *Relative eigensets*

Consider a system Σ with the nominal representation $y_{k+1} = f(y_k, u_k)$ and a model M with the nominal representation $\zeta_{k+1} = \varphi(\zeta_k, v_k)$. Using the recursion functions f and φ, we construct the function

$$(f, \varphi) : (R^p \times R^m)^2 \to (R^p)^2 : (y, s, \zeta, w) \mapsto (f(y, s), \varphi(\zeta, w)) \qquad (3.1.1)$$

which simultaneously represents the two systems Σ and M.

On the domain of the function (f, φ) we introduce the projections

$$\Pi_{y\zeta} : (R^p \times R^m)^2 \to (R^p)^2 : (y, s, \zeta, w) \mapsto (y, \zeta)$$

$$\Pi_{y-\zeta} : (R^p \times R^m)^2 \to R^p : (y, s, \zeta, w) \mapsto y - \zeta;$$

$$\Pi_{y\zeta w} : (R^p \times R^m)^2 \to R^p \times R^p \times R^m : (y, s, \zeta, w) \mapsto (y, \zeta, w)$$

$$\Pi_{ys\zeta} : (R^p \times R^m)^2 \to R^p \times R^m \times R^p : (y, s, \zeta, w) \mapsto (y, s, \zeta)$$

Next, let $\delta > 0$ be a real number, let $q > 0$ be an integer, and let z be a point in R^q. Denote by

$$N_\delta(z) := \{\zeta \in R^q : |\zeta - z| \leq \delta\}$$

the closed neighbourhood of radius δ around the point z. For a subset $S \subset R^q$, denote by

$$N_\delta(S) := \cup_{z \in S} N_\delta(z)$$

We are now ready to define the basic notion of our discussion, generalizing the concept of eigenset introduced in Hammer (1989, 1998).

(3.1.2) **Definition:** Let $f, \varphi : R^p \times R^m \to R^p$ be two functions, let $\delta,\ \Delta > 0$ be real numbers satisfying $\Delta > 2\delta$, and let (f, φ) be as in (3.1.1). Then, a non-empty subset $S \subset (R^p \times R^m)^2$ is a *(δ, Δ)-eigenset of f relative to φ* if the following conditions hold:

(i) $|\Pi_{y-\zeta}[S]| \leq \Delta - 2\delta$, and
(ii) $(f, \varphi)[N_\delta(S)] \subset \Pi_{y\zeta}[S]$.

The number δ is called the *contraction radius* of the relative eigenset S.

To discuss the intuitive meaning of Definition (3.1.2), consider a (δ, Δ)-eigenset S of f relative to φ. Since S is a subset of $(R^p \times R^m)^2$, each point of S can be regarded as a twosome $((y, s), (\zeta, w))$ of state-input pairs, where (y, s) is a state-input pair of Σ and (ζ, w) is a state-input pair of M. In this way, S induces a correspondence among such pairs, where (y, s) corresponds to (ζ, w) if $(y, s, \zeta, w) \in S$. Now, let (y, s) and (ζ, w) be such a corresponding pair. Then, condition (i) of Definition 3.1.2 means that the discrepancy between y (the state of Σ) and ζ (the corresponding state of M) does not exceed Δ, even if independent disturbances of amplitude not exceeding δ are added to y and to ζ.

Condition (ii) of Definition (3.1.2) indicates that S is a conditional invariant subset of the function (f, φ). Conditional invariant subsets have played an important role in the investigation of non-linear dynamical systems (Lasalle and Lefschetz 1961, Lefschetz 1965), and they serve as a conceptual foundation for the notion of controlled invariant subspaces and controllability subspaces in linear control theory (Wonham and Morse 1970, Wonham 1974). The notion of conditional invariant subset is also critical to our present discussion.

To further discuss the significance of condition (ii) of Definition (3.1.2), consider a pair $(y, \zeta) \in \Pi_{y\zeta}S$; recall that y is a state value of Σ and ζ is a state value of M. Since $(y, \zeta) \in \Pi_{y\zeta}S$, there are input values s and w such that $(y, s, \zeta, w) \in S$. Assume now that Σ is at the state y and M is at the state ζ; apply the input value s to Σ and the input value w to M. Let y^+ and ζ^+ denote the next states of Σ and M, respectively. Then, condition (ii) of Definition (3.1.2) implies that $(y^+, \zeta^+) \in \Pi_{y\zeta}S$ (which is the invariance property of the set). In view of condition (i) of Definition (3.1.2), this implies that the discrepancy between y^+ and ζ^+ does not exceed Δ. In other words, using the input values s and w indicated by S, allows us to maintain a discrepancy not exceeding Δ for the next step.

When repeated step after step, the process of the previous paragraph allows us to construct corresponding input sequences of Σ and of M that maintain a discrepancy of Δ or less among the trajectories of the two systems. We show later that the resulting input sequence of Σ can be created by a feedback controller. This allows us to create a Δ-approximant of M by combining a feedback controller with the system Σ. The feedback controller can be calculated from a (δ, Δ)-eigenset S of f relative to φ. In its turn, a (δ, Δ)-eigenset S of f relative to φ can be calculated from the solution of a set of algebraic inequalities induced by the given recursion functions f and φ (see § 4). In this way we obtain a constructive solution of the approximate model matching problem. For this purpose, we shall need the following special kind of relative eigenset.

(3.1.3) **Definition:** Let $f,\ \varphi : R^p \times R^m \to R^p$ be two functions, and let $\theta, \delta, \Delta > 0$ be real numbers, where $\Delta > 2\delta$. A (δ, Δ)-eigenset S of f relative to φ is *input complete* with amplitude θ if $S \supset (\Pi_{ys\zeta}[S]) \times (-(\theta + \delta), (\theta + \delta)]^m$.

An input complete eigenset S of f relative to φ has the special property that it includes all input values of φ of amplitude up to $\theta + \delta$; the addition of δ comes to account for the disturbance effects. To be more specific, let y be a state of Σ, let s be an input value of Σ, let ζ be a state of M, and let w be an input value of M. Then, for every triple $(y, s, \zeta) \in \Pi_{ys\zeta}S$, the entire set

$(y, s, \zeta) \times [-(\theta + \delta), (\theta + \delta)]^m$ is contained in S. This means that for any pair of states $(y, \zeta) \in \Pi_{y\zeta}S$ and for any input value $w \in [-(\theta + \delta), (\theta + \delta)]^m$ of M, there is an input value of s of Σ such that $(y, s, \zeta, w) \in S$. When this s is used as the input value of Σ (while w is the input value of M), the next states y^+ of Σ and ζ^+ of M do not differ by more than Δ. This allows us to achieve approximate model matching for all input sequences of M whose amplitude does not exceed θ. In the next subsection we show how these observations lead to the derivation of controllers.

3.2. *Deriving controllers*

In this subsection we describe the process that leads from an input-complete relative eigenset to an approximate model matching controller. The derivation of input-complete relative eigensets is described in § 4. We assume that the system Σ that needs to be controlled, as well as the model M that needs to be approximately matched, are given in terms of nominal state representations of the forms (1.0.3) and (1.0.4), respectively. The controllers we derive are of the form (1.0.6), so that the only quantity that needs to be calculated is the feedback function σ. The results can be applied to the more general case of recursive systems by using the notion of formal realization (§ 1.1).

Let $S \subset (R^p \times R^m)^2$ be a (δ, Δ)-eigenset of f relative to φ, input complete with amplitude $\theta > 0$. For each point $(y, \zeta, w) \in R^p \times R^p \times R^m$, we construct now a subset $U_S(y, \zeta, w) \subset R^m$, called the *feedback value set* of S. We show later that $U_S(y, \zeta, w)$ consists of values the feedback function σ can take when the system Σ is at the state y, the model M is at the state ζ, and the external input value is w.

(3.2.1) **Construction of the feedback value set $\mathbf{U_S}$**

(i) When $(y, \zeta, w) \in \Pi_{y\zeta w}[S]$:
The set $U_S(y, \zeta, w)$ consists of all points $s \in R^m$ satisfying $(y, s, \zeta, w) \in S$.

(ii) When $(y, \zeta, w) \notin \Pi_{y\zeta w}[S]$:
Let $A(y, \zeta, w)$ be the set of all points $(a, b, c) \in \Pi_{y\zeta w}[S]$ satisfying $|y - a| \leq 2\delta/3, |\zeta - b| \leq \delta$ and $|w - c| \leq \delta$. Then, we distinguish among three cases:

(a) If $A(y, \zeta, w) \neq \varnothing$, the set $U_S(y, \zeta, w)$ consists of all points $s \in R^m$ satisfying $(a, s, b, c) \in S$ for some vector $(a, b, c) \in A(y, \zeta, w)$.

(b) If $(y, \zeta, w) \in N_\delta(\Pi_{y\zeta w}[S])$ and $A(y, \zeta, w) = \varnothing$, the set $U_S(y, \zeta, w)$ consists of all vectors $s \in R^m$ such that $(a, s, b, c) \in S$ for some vector $(a, b, c) \in \Pi_{y\zeta w}[S]$ satisfying $|(a, b, c) - (y, \zeta, w)| \leq \delta$.

(c) If $(y, \zeta, w) \notin N_\delta(\Pi_{y\zeta w}[S])$, then $U_S(y, \zeta, w) := 0$ is the set consisting of the zero vector alone.

To examine the qualitative significance of the feedback value set $U_S(y, \zeta, w)$, consider the closed loop system (1.0.2) with the controller C of (1.0.6). Assume for a moment that all disturbances and errors are zero. Let y_0 be the initial condition of Σ, let ξ_0 be the initial condition of M, let ζ_0 be the initial condition of the controller C of (1.0.6), and let $v \in S(\theta^m)$ be the external input sequence of the closed loop system. Since all disturbances are zero, the initial conditions satisfy $\xi_0 = \zeta_0$, and the input sequence of (1.0.2) satisfies $w = v$. This implies that the output sequence ξ of M is equal to the sequence ζ generated within the controller C of (1.0.6).

Consider the case where Σ and M start from initial conditions within the eigenset S, so that $(y_0, \zeta_0) \in \Pi_{y\zeta}[S]$. Assume that the controller C of (1.0.2) is constructed so that the sequence s it generates satisfies

$$s_k \in U_S(y_k, \zeta_k, w_k) \tag{3.2.2}$$

$k = 0, 1, 2, \ldots$. We show now by induction that under these circumstances

$$(y_k, \zeta_k) \in \Pi_{y\zeta}[S] \qquad \text{for all integers } k \geq 0 \tag{3.2.3}$$

Indeed, assume that $(y_i, \zeta_i) \in \Pi_{y\zeta}[S]$ for some integer $i \geq 0$. (Note that this is valid for $i = 0$ by our choice of initial conditions.) Then, since S is input complete and $w_i = v_i \in [-\theta, \theta]^m$, it follows that $(y_i, \zeta_i, w_i) \in \Pi_{y\zeta w}[S]$. Further, by (3.2.1)(i), the value $s_i \in U_S(y_i, \zeta_i, w_i)$ satisfies $(y_i, s_i, \zeta_i, w_i) \in S$. But then, since $(y_{i+1}, \zeta_{i+1}) = (f, \varphi)(y_i, s_i, \zeta_i, w_i)$, part (ii) of Definition (3.1.2) implies that $(y_{i+1}, \zeta_{i+1}) \in \Pi_{y\zeta}[S]$, and (3.2.3) follows by induction.

As a direct consequence of (3.2.3) and part (i) of Definition (3.1.2), we obtain that

$$|y_k - \zeta_k| \leq \Delta - 2\delta$$

for all integers $k \geq 0$. Thus, the sequence s generated according to (3.2.2) drives Σ so as to maintain a discrepancy of less than Δ among the output sequences of Σ and of M. In other words, the feedback value set U_S forms the basis for a solution to the approximate model matching problem.

Most importantly, (3.2.2) represents a feedback rule for generating the sequence s. To see that this is the case, define a function $\sigma : R^p \times R^p \times R^m \to R^m : (y, \zeta, w) \mapsto \sigma(y, \zeta, w)$ by setting

$$\sigma(y, \zeta, w) := s \tag{3.2.4}$$

where s is a point of the set $U_S(y, \zeta, w)$. Our earlier discussion implies then that with this σ, the controller C of (1.0.6) achieves the design objective (2.1.4), at least in the nominal case. The next statement, which is one of the main results of the present paper, shows that the same controller solves the approximate model matching problem in the presence of disturbances as well.

(3.2.5) **Theorem:** *Let Σ, $M : S(R^m) \to S(R^p)$ be input/state systems having the recursion functions $f, \varphi : S(R^p) \times S(R^m) \to S(R^p)$ and the initial conditions y_0 and ξ_0, respectively. Assume that there is a (δ, Δ)-eigenset S of f relative to φ which is input complete with amplitude $\theta > 0$, and that the initial conditions satisfy $(y_0, \xi_0) \in \Pi_{y\zeta}[S]$. Let $U_S(\cdot)$ be the feedback value set induced by S, and build a function $\sigma : R^p \times R^p \times R^m \to R^m$ by setting $\sigma(y, \zeta, w) := s$, where s is a point of $U_S(y, \zeta, w)$. Then, with this choice of σ, the controller C of* (1.0.6) *solves the approximate model matching problem* (2.1.4), *as long as the disturbance amplitudes satisfy $n_0 \le \delta$ and $n_1 \le \delta/3$.*

The proof of Theorem (3.2.5) is provided in § 5.

As can be seen from Theorem (3.2.5), the existence of a relative eigenset S is a sufficient condition for the existence of a controller C that turns Σ into a Δ-approximant of the model M. Furthermore, the theorem also indicates an explicit construction of the controller C in terms of the relative eigenset S. We show in § 4 that an appropriate relative eigenset S can be calculated by solving a set of algebraic inequalities derived from the given recursion functions f and φ. Combined with the results of § 4, Theorem (3.2.5) provides a complete means for the design of controllers that solve the approximate model matching problem. Finally, note that although Theorem (3.2.5) is stated for the case of input/state systems, it can be used for general recursive systems by employing the notion of formal realization of § 1.1.

Regarding disturbances, Theorem (3.2.5) shows that a sufficient condition for the existence of a controller is that the disturbance amplitude bounds satisfy the inequalities

$$n_0 \le \delta \quad \text{and} \quad n_1 \le \delta/3 \tag{3.2.6}$$

where δ is the contraction radius of the relative eigenset S. This gives rise to the question of whether or not the validity of these amplitude bounds is also necessary for the existence of an approximate model matching controller. The next statement shows that (3.2.6) is indeed a necessary condition for the existence of an approximate model matching controller, for certain pairs of systems Σ and M. Thus, when considering the entire class of input/state systems, the disturbance amplitude bounds given by (3.2.6) are tight. This provides practical significance to the contraction radius δ of a relative eigenset: it is the factor that determines the largest disturbance amplitudes that can be tolerated by the approximate model matching controller.

(3.2.7) **Proposition:** *There is a pair of input/state systems Σ, $M : S(R^m) \to S(R^p)$ with recursion functions f and φ, respectively, for which the following are true:*

(i) *f has a (δ, Δ)-eigenset S relative to φ with complete input space of amplitude $\theta > 0$.*

(ii) *Any controller C that solves the approximate model matching problem* (2.1.4) *for Σ and M permits only disturbances with amplitude bounds satisfying* (3.2.6).

The proof of Proposition (3.2.7) is provided in § 5.

Our discussion so far indicates that relative eigensets are the critical quantity on which the solution to the approximate model matching problem depends: One can calculate a controller from a relative eigenset; and the contraction radius of the relative eigenset provides bounds for permissible disturbance amplitudes. Thus, it is important to develop techniques for the calculation of relative eigensets, and to discuss maximal contraction radii of relative eigensets. These topics are addressed in the next section.

4. Computation of relative eigensets

We turn now to the examination of techniques for the computation of relative eigensets. When combined with Theorem (3.2.5), these techniques allow us to construct controllers that solve the approximate model matching problem. As mentioned earlier, the notion of relative eigenset is a generalization of the notion of eigenset (Hammer 1989, 1998). Accordingly, computational techniques developed in Hammer (1998) for eigensets can be adapted to the case of relative eigensets. In particular, it was seen in Hammer (1998) that reachability is helpful for the calculation of eigensets. The next subsection shows that reachability is also helpful for the calculation of relative eigensets. This indicates a linkage between reachability and approximate model matching for non-linear systems.

4.1. *Reachability and the calculation of relative eigensets*

Consider a system Σ with the nominal recursive representation (1.0.3). Assume the system starts from the initial condition $y := y_0 \in R^p$, and is driven by the input sequence $(u_0, u_1, u_2, \ldots) \in S(R^m)$. The state y_i of the system at the step $i \ge 1$ is then given by the iterated recursion

$$y_i = f(f \ldots f(f(y, u_0), u_1) \ldots, u_{i-1}), i = 1, 2, \ldots$$

Introducing the shorthand notation

$$f^i(y, u_0, \ldots, u_{i-1}) := f(f \ldots f(f(y, u_0), u_1) \ldots, u_{i-1})$$

we can write

$$y_i = f^i(y, u_0, \ldots, u_{i-1})$$

A state $y' \in R^p$ is *reachable* from the state $y \in R^p$ in i steps if there is an input list $u_0, \ldots, u_{i-1}$ for which $f^i(y, u_0, \ldots, u_{i-1}) = y'$. The set of all states that are reachable from y in i steps is then given by

$$\operatorname{Im} f^i(y, \cdot) := \{f^i(y, u_0, \ldots, u_{i-1}) : u_0, \ldots, u_{i-1} \in R^m\}$$

The relaization (1.0.3) is *globally reachable* if there is an integer $n > 0$ for which the following is true: every state $y' \in R^p$ is reachable from every state $y \in R^p$ in n steps; i.e. if $\operatorname{Im} f^n(y, \cdot) = R^p$ for all $y \in R^p$. Hammer (1998, § 4) contains computational tests for the verification of global reachability; it is shown there that many systems of practical interest are globally reachable.

The realization (1.0.3) is *everywhere locally reachable* if there is an integer $q \geq 1$ for which the iterated function $f^q(y, \cdot)$ is an open function for all states $y \in R^p$. A computational test for local reachability is listed in Hammer (1998, Proposition 2); for this test, one can take $q = p$, where p is the dimension of the state space.

Let Σ be a system with the realization (1.0.3), and assume that Σ is globally reachable as well as everywhere locally reachable. Let n be the smallest integer satisfying (i) $\operatorname{Im} f^n(y, \cdot) = R^p$ for all $y \in R^p$, and (ii) $f^n(y, \cdot)$ is an open function for all states $y \in R^p$. Then, we call n the *reachability integer* of the system (see Hammer 1998 for a discussion). Although the notion of local reachability is not directly mentioned below, it is instrumental for proving the existence of some of the quantities, as seen in Hammer (1998, § 4).

Let then Σ be globally reachable with the reachability integer n, and consider for a moment the nominal case (where all disturbances are set to zero). Let ξ_0 be the initial condition of the model M, and let v be the input sequence of M. The output sequence $\xi_0, \xi_1, \xi_2, \ldots \in R^p$ of M is then given by the recursion $\xi_{k+1} = \varphi(\xi_k, v_k)$.

Assume that Σ starts from the initial condition $y_0 = \xi_0$, as does M. Then, using global reachability of Σ, we can construct an input sequence $u \in S(R^m)$ that drives Σ so that its output values match the output values of M at steps that are integer multiples of the reachability integer n. Indeed, we have $\xi_0 = y_0$; and global reachability implies that there is an input list $u_0(0), \ldots, u_{n-1}(0)$ such that

$$y_n = f^n(y_0, u_0(0), \ldots, u_{n-1}(0)) = \xi_n$$

Repeating the same process every n steps, we obtain for every integer $j \geq 0$ a list $u_0(j), \ldots, u_{n-1}(j)$ of input values for which

$$\begin{aligned} y_{(j+1)n} &= f^n(y_{jn}, u_0(j), \ldots, u_{n-1}(j)) \\ &= \xi_{(j+1)n} = \varphi^n(\xi_{jn}, v_{jn}, \ldots, v_{(j+1)n-1}) \end{aligned} \quad (4.1.1)$$

Concatenating these lists into the sequence

$$u = u_0(0), \ldots, u_{n-1}(0), u_0(1), \ldots, u_{n-1}(1), \ldots \quad (4.1.2)$$

yields an input sequence that drives Σ so that its response sequence y satisfies

$$y_{jn} = \xi_{jn} \quad (4.1.3)$$

for all integers $j \geq 0$. In other words, with this input sequence, the output values of Σ and of M are identical at all steps that are integer multiples of the reachability integer n. Note that by (4.1.1), this input sequence of Σ can be calculated from the known initial conditions and the known input sequence v of M. We show later that the sequence u can be generated by a feedback controller.

It is important to emphasize that although (4.1.3) can be satisfied in all cases when Σ is globally reachable, it is still possible that there is no input sequence u of Σ for which

$$|y_k - \xi_k| \leq \Delta - 2\delta \quad (4.1.4)$$

for all integers $k \geq 0$. Substantial divergence between the two trajectories y and ξ may occur at steps that are not integer multiples of the reachability integer n. Whether or not (4.1.4) can be achieved for all steps $k \geq 0$ depends, of course, on the dynamical properties of the systems Σ and M.

The notion of reachability helps transform the problem of finding the infinite sequence u into a problem of solving the n inequalities

$$\left.\begin{aligned} &|f^i(y_0, u_0, \ldots, u_{i-1}) - \xi_i| \leq \Delta - 2\delta, \quad i = 1, 2, \ldots, n-1 \\ &|f^n(y_0, u_0, \ldots, u_{n-1}) - \xi_n| = 0 \end{aligned}\right\} \quad (4.1.5)$$

Assume there is a solution $u_0(y_0, \xi_1), u_1(y_0, u_0, \xi_2), \ldots, u_{n-1}(y_0, u_0, \ldots, u_{n-2}, \xi_n)$ of (4.1.5), where we have made explicit the dependence of u_j on all relevant variables. Then, the entire sequence u can be assembled by using the concatenation (4.1.2) with the values

$$\left.\begin{aligned} u_0(j) &= u_0(y_{jn}, \xi_{jn+1}) = u_0(y_{jn}, \varphi(\xi_{jn}, v_{jn})) \\ u_i(j) &:= u_i(y_{jn}, \mu_{jn}, \ldots, u_{jn+i-1}, \xi_{jn+i+1}) \\ &= u_i(y_{jn}, u_{jn}, \ldots, u_{jn+i-1}, \varphi(\xi_{jn+i}, v_{jn+i})) \end{aligned}\right\} \quad (4.1.6)$$

$i = 1, \ldots, n-1, j = 0, 1, 2, \ldots$. This process divides the infinite sequence u into segments of length n, allowing a finite dimensional computation for each step.

The requirement (4.1.3) is overly strict, as it does not allow for errors at steps that are integer multiples of n;

(4.1.3) needs to be weakened by permitting some discrepancy among the values of y_{jn} and ξ_{jn}. For this purpose we introduce a design parameter given by the real number $\rho > 0$ and replace (4.1.3) by the requirement

$$|y_{jn} - \xi_{jn}| \leq \rho, \qquad j = 0, 1, 2, \dots \tag{4.1.7}$$

We assume that (4.1.7) is also valid for the initial conditions, i.e. for the case $j = 0$. Since disturbances of amplitude δ can be added to y and to ξ, the restriction

$$\rho + 2\delta \leq \Delta \tag{4.1.8}$$

is needed to guarantee that the total discrepancy never exceeds Δ. From (2.1.1) and (4.1.7) we obtain

$$|\xi_{jn}| \leq \Theta, \qquad |y_{jn}| \leq \Theta + \rho \tag{4.1.9}$$

for all integers $j \geq 1$.

To conform with common practical issues, we require all input sequences of the system Σ to be bounded by a specified amplitude bound $\mu > 0$. The value of μ is determined by the physical characteristics of Σ.

We now turn to a detailed description of the process of calculating relative eigensets. It is convenient to introduce at this point some projections that are utilized in the ensuing discussion.

$$\Pi_i : (R^m)^j \to (R^m)^{i+1} : (u_0, \dots, u_{j-1}) \mapsto (u_0, \dots, u_i)$$

$$\pi_i : (R^m)^j \to R^m : (u_0, \dots, u_{j-1}) \mapsto u_i$$

where $0 \leq i \leq j - 1$. Our goal at this point is to achieve the following objective, as a step toward deriving a solution to the approximate model matching problem.

(4.1.10) **Objective:** Satisfy the requirements (4.1.4) and (4.1.7), while allowing disturbances of amplitude not exceeding δ to be added to y and to ξ.

Let ξ_0 and y_0 be initial conditions of M and Σ, and let $w_0, \dots, w_{n-1}$ and $s_0, \dots, s_{n-1}$ be input lists of M and Σ, respectively. As mentioned earlier, the nominal values satisfy $|\xi_0| \leq \Theta, |y_0| \leq \Theta + \rho$, $s_0, \dots, s_{n-1} \in [-\mu, \mu]^m$, and $w_0, \dots, w_{n-1} \in [-\theta, \theta]^m$, so all quantities are bounded. For an integer $i \in \{1, 2, \dots, n-1\}$, construct recursively a subset $(f, \varphi)^i_\delta(y_0, \xi_0, s_0, \dots, s_{i-1}, w_0, \dots, w_{i-1})$ of $R^p \times R^p$ as follows

$$(f, \varphi)^0_\delta := (y_0, \xi_0); \qquad \text{and}$$

$$\begin{aligned} &(f, \varphi)^k_\delta(y_0, \xi_0, s_0, \dots, s_{k-1}, w_0, \dots, w_{k-1}) : \\ &= (f, \varphi)[N_\delta((f, \varphi)^{k-1}_\delta(y_0, \xi_0, s_0, \dots, s_{k-2}, w_0, \dots, w_{k-2})), \\ &\qquad\qquad N_\delta(s_{k-1}), N_\delta(w_{k-1})] \end{aligned}$$

$k = 1, \dots, i$. In intuitive terms, the set $(f, \varphi)^i_\delta(y_0, \xi_0, s_0, \dots, s_{i-1}, w_0, \dots, w_{i-1})$ consists of pairs (x, ζ) where x is a state of Σ and ζ is a state of M, that can be reached at the step i under the following conditions:

(a) The system Σ starts from the nominal initial condition y_0 and is driven by the nominal input list $s_0, \dots, s_{i-1}$, while at each step (including the initial step) the state value as well as the input value are disturbed by a disturbance of amplitude not exceeding δ.

(b) The system M starts from the nominal initial condition ξ_0 and is driven by the nominal input list $w_0, \dots, w_{i-1}$, while at each step (including the initial step) the state value as well as the input value are disturbed by a disturbance of amplitude not exceeding δ.

Now, for fixed intial conditions y_0 and ξ_0, and for a fixed input list $w_0, \dots, w_{n-1}$ of M, let $W(y_0, \xi_0, w_0, \dots, w_{n-1})$ be the set of all input lists $s_0, \dots, s_{n-1} \in [-\mu, \mu]^m$ of Σ for which the following hold

$$|\Pi_{y-\xi}[(f, \varphi)^i_\delta(y_0, \xi_0, s_0, \dots, s_{i-1}, w_0, \dots, w_{i-1})]| \leq \Delta - 2\delta \tag{4.1.11}$$

for all $i - 1, \dots, n-1$; and

$$|\Pi_{y-\xi}[(f, \varphi)^n_\delta(y_0, \xi_0, s_0, \dots, s_{n-1}, w_0, \dots, w_{n-1})]| \leq \rho \tag{4.1.12}$$

Note that relations (4.1.11) and (4.1.12) represent a finite set of inequalities based on the two given recursion functions f and φ. The solution set $W(y_0, \xi_0, w_0, \dots, w_{n-1})$ of these inequalities forms the basis of our construction of a (δ, Δ)-eigenset of f relative to φ. We assume that the reachability properties of the system Σ are such that

$$W(y_0, \xi_0, w_0, \dots, w_{n-1}) \neq \varnothing \tag{4.1.13}$$

for all $y_0, \xi_0, w_0, \dots, w_{n-1}$ satisfying $|y_0| \leq \Theta + \rho$, $|\xi_0| \leq \Theta$, and $w_0, \dots, w_{n-1} \in [-\theta, \theta]^m$; this guarantees the existence of input sequences for which (4.1.4) and (4.1.7) hold.

The calculation of the set $W(y_0, \xi_0, w_0, \dots, w_{n-1})$ from the inequalities (4.1.11) and (4.1.12) may involve non-causal operations, since, for instance, the value s_i for $i < n - 1$ may depend on w_{n-1}. To guarantee causality, further restrictions must be imposed, as follows. For $n > 1$, define the set $V_i(y_0, \xi_0, w_0, \dots, w_i)$ by

$$\begin{aligned} &V_i(y_0, \xi_0, w_0, \dots, w_i) : \\ &\quad = \bigcap_{w_{i+1}, \dots, w_{n-1} \in [-\theta, \theta]^m} \prod\nolimits_i W(y_0, \xi_0, w_0, \dots, w_{n-1}), \\ &\qquad\qquad\qquad i = 0, \dots, n-2 \end{aligned} \tag{4.1.14}$$

Then, $V_i(y_0, \xi_0, w_0, \dots, w_i)$ depends only on the input values $w_0, \dots, w_i$ (and the initial conditions y_0, ξ_0), and whence can be calculated from information available at the step i.

Next, define the set $U(y_0, \xi_0, w_0, \ldots, w_{n-1})$ as the set of all lists $(s_0, \ldots, s_{n-1}) \in W(y_0, \xi_0, w_0, \ldots, w_{n-1})$ satisfying

$$\Pi_i(s_0, \ldots, s_{n-1}) \in V_i(y_0, \xi_0, w_0, \ldots, w_i), i = 0, \ldots, n-2$$

The set $U(y_0, \xi_0, w_0, \ldots, w_{n-1})$ consists then of input lists $s_0, \ldots, s_{n-1}$ that can be calculated causally; namely, for each $i \in \{0, \ldots, n-1\}$, the values $s_0, \ldots, s_i$ can be obtained from information available at the step i.

From our construction so far it follows that Objective (4.1.10) can be met if

$$U(y_0, \xi_0, w_0, \ldots, w_{n-1}) \neq \varnothing$$

for all $|\xi_0| \le \Theta$, for all $|y_0| \le \Theta + \rho$, and for all $w_0, \ldots, w_{n-1} \in [-\theta, \theta]^m$. We refer to $U(y_0, \xi_0, w_0, \ldots, w_{n-1})$ as the *causal solution set* of (4.1.11) and (4.1.12).

Assuming that $U(y_0, \xi_0, w_0, \ldots, w_{n-1}) \neq \varnothing$, we construct the sets

$$S_i := \left\{ (y, s, \xi, w) \;\middle|\; \begin{array}{l} (y, \xi) = (f, \varphi)^i_\delta(y_0, \xi_0, s_0, \ldots, s_{i-1}, w_0, \ldots, w_{i-1}) \\ s = \pi_i U(y_0, \xi_0, w_0, \ldots, w_{n-1}) \\ w \in [-\theta, \theta]^m \\ |\xi_0| \le \Theta \\ |y_0 - \xi_0| \le \rho \\ s_0, \ldots, s_{i-1} = \prod_{i-1} U(y_0, \xi_0, w_0, \ldots, w_{n-1}) \\ w_0, \ldots, w_{n-1} \in [-\theta, \theta]^m \end{array} \right\}$$

$i = 0, \ldots, n-1$. It follows then by the definition of $(f, \varphi)^i_\delta$, by (4.1.12), and by (4.1.9) that

$$(f, \varphi)[N_\delta(S_i)] \subset \Pi_{y\xi}[S_{i+1}], \qquad i = 0, \ldots, n-2, \qquad \text{and}$$
$$(f, \varphi)[N_\delta(S_{n-1})] \subset \Pi_{y\xi}[S_0]$$

so that we have a cyclical situation. Thus, the set

$$S := \bigcup_{i=0,\ldots,n-1} S_i \tag{4.1.15}$$

has the property

$$(f, \varphi)[N_\delta(S)] \subset \Pi_{y\xi}[S] \tag{4.1.16}$$

and S is a conditional invariant subset for the function (f, φ). Furthermore, (4.1.16) indicates that the conditional invariance of S is not destroyed by disturbances of amplitude not exceeding δ. In this way, the concept of reachability allowed us to build S by considering only n steps of the systems M and Σ.

Further, (4.1.11), (4.1.12) and (4.1.8) also imply that $|\Pi_{y-\xi}[S_i]| \le \Delta - 2\delta$ for all $i = 0, \ldots, n-1$, so that

$$|\Pi_{y-\xi}[S]| \le \Delta - 2\delta \tag{4.1.17}$$

Combining (4.1.16) and (4.1.17), it follows that S is a (δ, Δ)-eigenset of f relative to φ. Thus, we have obtained a constructive procedeure for the derivation of relative eigensets. As we can see, the critical step in this procedure is the solution of (4.1.11) and (4.1.12): a set of algebraic inequalities determined by the given recursion functions f and φ. The following statement summarizes our discussion in this regard.

(4.1.18) **Theorem:** *Let Σ and M be systems with the recursive representations $y_{k+1} = f(y_k, u_k)$ and $\xi_{k+1} = \varphi(\xi_k, w_k)$, respectively, where M is output bounded by the real number $\Theta > 0$. Let $U(y_0, \xi_0, w_0, \ldots, w_{n-1})$ be the causal solution set of* (4.1.11) *and* (4.1.12). *Assume there are real numbers $\Delta > 0, \delta > 0, \rho > 0, \theta > 0$, where $\rho + 2\delta \le \Delta$, for which $U(y_0, \xi_0, w_0, \ldots, w_{n-1}) \neq \varnothing$ for all $y_0, \xi_0, w_0, \ldots, w_{n-1}$ satisfying $|\xi_0| \le \Theta$, $|y_0| \le \Theta + \rho$, $|y_0 - \xi_0| \le \rho$ and $w_0, \ldots, w_{n-1} \in [-\theta, \theta]^m$. Then, the set S of* (4.1.15) *is a (δ, Δ)-eigenset of f relative to φ, input complete with amplitude θ.*

Once a (δ, Δ)-eigenset S of f relative to φ has been derived, we can build a controller that solves the approximate model matching problem by following the procedure of Theorem (3.2.5). As Theorem (3.2.5) suggests, the contraction radius δ of S determines disturbance amplitudes that the closed loop system is guaranteed to tolerate. Consequently, when solving the inequalities (4.1.11) and (4.1.12), it is beneficial to find the largest value of δ for which a solution exists.

(4.1.19) **Example:** Consider the model M given by the system

$$\xi_{k+1} = 0.5\xi_k + w_k$$

so that $\varphi(\xi, w) = 0.5\xi + w$ in this case. Suppose that the initial condition of M satisfies $|\xi_0| \le 1$ and the input amplitude bound is $\theta = 1$. It follows readily from well known properties of linear systems that M is BIBO-stable, and one can use the output bound $\Theta = 2$.

Let the system Σ that needs to be controlled be given by

$$y_{k+1} = [(y_k)^2 + 1]s_k$$

so that $f(y, s) = (y^2 + 1)s$ here. A slight reflection shows that Σ is globally as well as locally reachable, and its reachability integer is $n = 1$.

Assume that the specified discrepancy bound is $\Delta = 1$, and that $\rho = 1/2$. The objective is to find a controller, as well as a maximal value for the contraction radius δ. Note that according to (4.1.8), we must have $\delta \le 1/4$.

Now, the function (f, φ) here is given by

$$(f, \varphi)(y, \zeta, s, w) = ((y^2 + 1)s, 0.5\zeta + w)$$

Inequalities (4.1.11) and (4.1.12) reduce to a single inequality here (since $n = 1$), given by

$$[(y + \alpha)^2 + 1](s + \beta) - [0.5(\zeta + \gamma) + (w + \varepsilon)] \le \rho = 1/2 \tag{4.1.20}$$

where α, β, γ and ε represent disturbances. Note that (4.1.20) has to be valid for all

$$|\alpha| \le \delta, \quad |\beta| \le \delta, \quad |\gamma| \le \delta, \quad |\varepsilon| \le \delta$$
$$|\zeta| \le 2, \quad |w| \le 1, \quad |y - \zeta| \le 1/2 \qquad (4.1.21)$$

The amplitude bound on M implies that $|\zeta| \le 2$, so the inequality $|y - \zeta| \le 1/2$ implies that $|y| \le 2.5$ in the nominal case. We use the controller template (1.0.6).

We first find a feedback function σ that minimizes the discrepancy among the responses of the controlled system and the model M in the nominal case. For this function σ, we calculate the maximal value of δ for which the discrepancy between the responses of the controlled system and the model M does not exceed $\Delta = 1$. Direct observation shows that in this case one can achieve a zero nominal discrepancy between the responses of the controlled system and the model M by using the feedback function

$$s = \sigma(y, \zeta, w) = \frac{0.5\zeta + w}{y^2 + 1}$$

Substituting this value into (4.1.20), we obtain the inequality

$$[(y+\alpha)^2 + 1]\left(\frac{0.5\zeta + w}{y^2+1} + \beta\right)$$
$$-[0.5(\zeta + \gamma) + (w + \varepsilon)] \le 1/2$$

which has to be valid for all quantities satisfying (4.1.21). Reorganizing terms we obtain

$$[2y\alpha + \alpha^2]\left(\frac{0.5\zeta + w}{y^2+1}\right) + [(y+\alpha)^2 + 1]\beta$$
$$-[0.5\gamma + \varepsilon] \le 1/2 \qquad (4.1.22)$$

A slight reflection shows that the worst case of (4.1.22) subject to (4.1.21) occurs for $\alpha = \delta$, $\beta = \delta$, $\gamma = -\delta$, $\varepsilon = -\delta$, $\zeta = 2$, and $w = 1$. Substituting these values into (4.1.22) we obtain the inequality

$$[2y\delta + \delta^2]\left(\frac{2}{y^2+1}\right) + [(y+\delta)^2 + 1]\delta + 1.5\delta \le 1/2$$

A numerical calculation then shows that in order for this inequality to hold for all nominal values $|y| \le 2.5$, one must have approximately $\delta \lesssim 0.048$. Thus, the nominal feedback controller in this case is given by

$$C: \begin{cases} \zeta_{k+1} = 0.5\zeta_k + v_k \\ s_k = \sigma(y_k, \zeta_k, v_k) = \dfrac{0.5\zeta_k + v_k}{(y_k)^2 + 1} \end{cases}$$

while the maximal contraction radius is approximately $\delta \approx 0.048$.

As we can see from the example, the largest value of the contraction radius δ can be calculated directly in each case. This value of δ determines the largest permissible disturbance amplitudes compatible with Theorem (3.2.5), in the sense of Proposition (3.2.7).

By an argument similar to the one used to prove Hammer (1998, Theorem 4), the following statement can be shown to be true. When the system Σ is globally as well as locally reachable, and when the model M is output bounded, there are real numbers $\Delta, \delta, \rho, \theta > 0$ for which the relative eigenset S of (4.1.15) is not empty. When this is combined with Theorem (3.2.5), it implies that every recursive system whose formal realization is reachable, can be robustly stabilized by a controller of the form (1.1.7); stability here is in the BIBO sense.

To conclude, we have presented a rather general theory for the design of controllers for non-linear recursive systems. For the case of systems whose state is provided as output, the controllers use state feedback; for systems whose output is not a state, the controllers use output feedback, and no observers are involved. The theory is computational, and the calculation of controllers is based on the solution of a set of algebraic inequalities. As can be seen from the basic controller template (1.0.6), the controllers used in this framework ae always BIBO-stable (when the model M is BIBO-stable). The results presented here also provide a general computational methodology for the stabilization of non-linear recursive systems.

(4.1.23) **Remark:** The causality condition (4.1.14) is a relatively strict requirement, which may limit the fidelity by which Σ can follow the model M. In some applications, the causality requirement can be eliminated by approximating a delayed version of the model M, as follows. Let D denote the on-step delay operator. Then, one could require the closed loop system Σ_c to approximate the delayed model $D^{n-1}M$, rather than approximating M. This would eliminate the need of imposing (4.1.14), since the $(n-1)$-step delay has the effect of replacing $W(y_0, \xi_0, w_0, \ldots, w_{n-1})$ by $W(y_0, \xi_0, w_{-n+1}, \ldots, w_0)$, so that only past input values of M are involved. This, of course, changes the design requirements, and is only possible in applications where delays are non-disruptive.

5. Proofs and technicalities

The present section contains the proofs of Theorem (3.2.5) and Proposition (3.2.7).

Proof (of Theorem 3.2.5): We use the notation of Theorem (3.2.5), and assume that the disturbance amplitude bounds satisfy $n_0 \le \delta$ and $n_1 \le \delta/3$. Also, by the assumptions of Theorem (3.2.5), the initial conditions satisfy $(y_0, \xi_0) \in \Pi_{y\xi}[S]$. Let ζ_0 be the initial condition of the controller C of (1.0.6), and note the sequence $\zeta \in S(R^p)$ generated inside C. We can write

$\zeta_0 = \xi_0 + \omega_0$, where $|\omega_0| \le \delta$. Further, let $s \in S(R^m)$ be the output sequence of the controller C as generated by the assignment (3.2.4). The response sequence $y \in S(R^p)$ of the closed loop system is then given by the recursion

$$y_{k+1} = f(y_k, s_k + v_{1,k}) + v_{2,k} \tag{5.0.1}$$

where s_k satisfies (3.2.2). Recall that, according to (2.1.2), our objective is to approximate the response of the disturbed model M having the representation

$$\xi_{k+1} = \varphi(\xi_k + v_{5,k-1}, v_k + v_{4,k}) + v_{5,k} \tag{5.0.2}$$

we set here $v_{5,-1} := \omega_0$ to accommodate the initial condition error. This then yields $\xi = \zeta$, where ζ is the sequence generated by the implementation of M inside the controller C of (1.0.6).

At the step $k+1$, the discrepancy between the output value y_{k+1} of the closed loop system Σ_c and the output value ξ_{k+1} of the disturbed model M is given by

$$|y_{k+1} - \xi_{k+1}| = |y_{k+1} - \zeta_{k+1}| = |[f(y_k, s_k + v_{1,k}) + v_{2,k}] - [\varphi(\zeta_k + v_{5,k-1}, v_k + v_{4,k}) - v_{5,k}]|$$

where

$$s_k = \sigma(y_k + v_{3,k}, \zeta_k, v_k + v_{4,k})$$

We need to show that under the assumptions of Theorem (3.2.5), one has

$$|y_{k+1} - \xi_{k+1}| \le \Delta \tag{5.0.3}$$

for all integers $k = 0, 1, 2, \ldots$.

To that end, let $k \ge 0$ be an integer, and let (a_k, b_k, v_k) be a point in the set $\Pi_{y\xi w}[S]$. Consider the recursion

$$\left.\begin{aligned} a_{k+1} &= f(a_k + v_{2,k-1}, s_k + v_{1,k}) \\ b_{k+1} &= \varphi(b_k + v_{5,k-1}, v_k + v_{4,k}) \\ s_k &= \sigma(a_k + v_{2,k-1} + v_{3,k}, b_k + v_{5,k-1}, v_k + v_{4,k}) \end{aligned}\right\} \tag{5.0.4}$$

Refer to the point

$$(z_k, \zeta_k, w_k) := (a_k + v_{2,k-1} + v_{3,k}, b_k + v_{5,k-1}, v_k + v_{4,k}) \tag{5.0.5}$$

which is the argument of σ in (5.0.4). Then, $|z_k - a_k| \le 2n_1 \le 2\delta/3$ and $|(z_k, \zeta_k, w_k) - (a_k, b_k, v_k)| \le \max\{2n_1, n_0, n_0\} \le \delta$ under our assumptions. Consequently, the set $A(\cdot)$ of (3.2.1) (ii) satisfies $A(z_k, \zeta_k, w_k) \ne \varnothing$. Since

$$s_k = \sigma(z_k, \zeta_k, w_k) \in U_S(z_k, \zeta_k, w_k)$$

it follows by (3.2.1) that there is a point $(\alpha, \beta, \gamma) \in A(z_k, \zeta_k, w_k)$ such that

$$(\alpha, s_k, \beta, \gamma) \in S \tag{5.0.6}$$

here, by the definition of the set A, one has

$$|z_k - \alpha| \le 2\delta/3, \quad |\zeta_k - \beta| \le \delta \quad \text{and} \quad |w_k - \gamma| \le \delta \tag{5.0.7}$$

so that $(z_k, s_k, \zeta_k, w_k) \in N_\delta(S)$.

Using (5.0.7) together with the equalities $z_k = a_k + v_{2,k-1} + v_{3,k}$, $\zeta_k = b_k + v_{5,k-1}$, and $w_k = v_k + v_{4,k}$ implied by (5.0.5), we obtain

$$\begin{aligned} &|(a_k + v_{2,k-1}, s_k + v_{1,k}, b_k + v_{5,k-1}, v_k + v_{4,k}) - (\alpha, s_k, \beta, \gamma)| \\ &= \max\{|a_k + v_{2,k-1} - \alpha|, |v_{1,k}|, |\zeta_k - \beta|, |w_k - \gamma|\} \\ &\le \max\{|[a_k + v_{2,k-1} - z_k] - [\alpha - z_k]|, \delta\} \\ &= \max\{|[-v_{3,k}] - [\alpha - z_k]|, \delta\} \\ &\le \max\{|v_{3,k}| + |\alpha - z_k|, \delta\} \\ &\le \max\{\delta/3 + 2\delta/3, \delta\} = \delta \end{aligned}$$

where the last inequality is by (5.0.7). Since $(\alpha, s_k, \beta, \gamma) \in S$ according to (5.0.6), this shows that

$$(a_k + v_{2,k-1}, s_k + v_{1,k}, b_k + v_{5,k-1}, v_k + v_{4,k}) \in N_\delta(S)$$

But then, using the fact that S is a (δ, Δ)-eigenset of f relative to φ, it follows that (a_{k+1}, b_{k+1}) of (5.0.4) satisfies $(a_{k+1}, b_{k+1}) \in \Pi_{y\xi}[S]$.

By (5.0.1), (5.0.4) and (5.0.2), we have $y_{k+1} = a_{k+1} + v_{2,k}$ and $\xi_{k+1} = \zeta_{k+1} = b_{k+1} + v_{5,k}$. Since $|\Pi_{y-\xi}S| \le \Delta - 2\delta$ we have $|a_{k+1} - b_{k+1}| \le \Delta - 2\delta$; and since $|v_2| \le \delta/3$, and $|v_{5,k-1}| \le \delta$, we obtain

$$|y_{k+1} - \xi_{k+1}| = |a_{k+1} + v_{2,k} - (b_{k+1} + v_{5,k})| \le |a_{k+1} - b_{k+1}| + |v_{2,k}| + |v_{5,k}| < \Delta$$

Finally, in view of the fact that the initial conditions satisfy $(y_0, \xi_0) \in \Pi_{y\xi}[S]$, it follows by induction that (5.0.3) is valid for all integers $k \ge 0$. This concludes our proof. □

Next, we provide the proof of Proposition (3.2.7).

Proof (of Proposition 3.2.7): The proof consists of constructing an example for which the statement is true. The example we provide is of single-input single-output systems $M, \Sigma : S(R) \to S(R)$; we take the model M to be the zero system, while the system Σ has the nominal recursive representation.

$$y_{k+1} = f(y_k, u_k) := \mu[0.1 - \max(|y_k|, |u_k|)] + \mu[0.1 - \max(|y_k - \alpha|, |u_k - \alpha|)] - 1$$

where μ denotes the unit step function, and $\alpha > 0.1$ is a fixed number (to be selected later). We take $\Delta = 0.3$, $\delta = 0.1$, $\rho = 0.1$, $\theta = 1$. A direct observation shows that the set

$$S = \{(0, 0, 0, [-1.1]), (\alpha, \alpha, 0, [-1.1])\}$$

is a (δ, Δ)-eigenset of f relative to the zero function here, and that $\delta = 0.1$ is the largest contraction radius possible in this case.

We assume that all disturbances have the same amplitude bound $\varepsilon > 0$, and we take $\alpha = 4\varepsilon$. Note that by Theorem (3.2.5), a possible value for ε is $\varepsilon = \delta/3$. We show below that in the present case $\varepsilon = \delta/3$ is indeed the maximal permissible value of ε, so that the disturbance amplitude bound of Theorem (3.2.5) is tight in this example.

To show that it is so, let $k \geq 0$ be an integer, and consider the particular case where Σ is at the nominal state $x_k = 0$, and the disturbance values are $\upsilon_{2,k-1} = \varepsilon$, $\upsilon_{3,k} = \varepsilon$. Recall that when disturbances are included, the recursive representation of Σ is given by (1.0.5), so that the state value y_k that appears in the argument of f for the step k is actually

$$y_k = x_k + \upsilon_{2,k-1} = 0 + \upsilon_{2,k-1} = \varepsilon$$

Using the notation of diagram (1.0.2), we have

$$z_k = y_k + \upsilon_{3,k} = 2\varepsilon$$

Now, z_k is the value received by the feedback function σ of (1.0.6) as an estimate of the state of Σ. Since the point $z_k = 2\varepsilon$ in the middle between the two possible states 0 and $\alpha = 4\varepsilon$ of S, there is no preference to any side. Suppose the assignment of σ was made so that $\sigma(2\varepsilon, 0, 0) := \alpha$, as if the nominal state was $x_k = \alpha$ (which is the wrong estimate in this case); then, $s_k = \alpha$. (If the assignment was $\sigma(2\varepsilon), 0, 0) := 0$, then a similar situation occurs when the nominal state is $x_k = \alpha$.)

Under these conditions, the arguments of f at the step k become $(y_k, s_k) = (\varepsilon, \alpha) = (\varepsilon, 4\varepsilon)$. In order not to exceed the discrepancy Δ, we must have then that $(\varepsilon, 4\varepsilon) \in \Pi_{yu} N_\delta(S)$, which in this case reduces to the requirement $(\varepsilon, 4\varepsilon) \in N_\delta\{(0,0), (4\varepsilon, 4\varepsilon)\}$. The largest value of $\varepsilon > 0$ that satisfies this requirement must be such that $4\varepsilon - \varepsilon \leq \delta$, or $3\varepsilon \leq \delta$. Thus, for the present example, the model matching problem can be solved (if and) only if $\varepsilon \leq \delta/3$, where δ is the contraction radius of the eigenset. This concludes our proof. □

References

BARAMOV, L., and KIMURA, H., 1995, Nonlinear coprime factorizations and parametrization of a class of stabilizing controllers. *Proceedings of the 1995 IEEE Conference on Decision and Control*, pp. 981–986.

CHEN, G., and DE FIGUEIREDO, R. J. P., 1990, Construction of the left coprime fractional representations for a class of nonlinear control systems. *Systems and Control Letters*, **14**, 353–361.

DESOER, C. A., and KABULI, M. G., 1988, Right factorization of a class of nonlinear systems. *IEEE Transactions on Automatic Control*, **AC-33**, 755–756.

HAMMER, J., 1984a, Nonlinear systems: stability and rationality. *International Journal of Control*, **40**, 1–35.

HAMMER, J., 1984b, On nonlinear systems, additive feedback, and rationality. *International Journal of Control*, **40**, 953–969.

HAMMER, J., 1985, Nonlinear systems, stabilization, and coprimeness. *International Journal of Control*, **42**, 1–20.

HAMMER, J., 1989a, Robust stabilization of nonlinear systems. *International Journal of Control*, **49**, 629–653.

HAMMER, J., 1989b, State feedback for nonlinear control systems. *International Journal of Control*, **50**, 1961–1980.

HAMMER, J., 1991, State feedback, confinement, and stabilization for nonlinear continuous time systems. *International Journal of Control*, **53**, 799–821.

HAMMER, J., 1994, Internally stable nonlinear systems with disturbances: a parametrization. *IEEE Transactions on Automatic Control*, **AC-39**, 300–314.

HAMMER, J., 1998, Handling disturbances in nonlinear control: the use of state feedback. *International Journal of Control*, **70**, 761–787.

LASALLE, J. F., and LEFSCHETZ, S., 1961, *Stability by Liapunov's Direct Method with Applications* (New York: Academic Press).

LEFSCHETZ, S., 1965, *Stability of Nonlinear Control Systems* (New York: Academic Press).

PAICE, A. D. B., and MOORE, J. B., 1990, Robust stabilization of nonlinear plants via left coprime factorizations. *Systems and Control Letters*, **15**, 121–129.

PAICE, A. D. B., and VAN DER SCHAFT, A. J., 1994, Stable kernel representation and the Youla parametrization for nonlinear systems. *Proceedings of the 33rd IEEE Conference on Decision and Control*, Lake Buena Vista, Florida, USA, pp. 781–786.

SONTAG, E. D., 1989, Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, **AC-34**, 435–444.

VERMA, M. S., 1988, Coprime fractional representations and stability of nonlinear feedback systems. *International Journal of Control*, **48**, 897–918.

VERMA, M. S., and HUNT, L. R., 1993, Right coprime factorization and stabilization for nonlinear systems. *IEEE Transactions on Automatic Control*, **AC-38**, 222–231.

WONHAM, W. M., 1974, Linear multivariable control: a geometric approach. *Lecture Notes in Economics and Mathematical Systems*, No. 101 (Berlin: Springer Verlag).

WONHAM, W. M., and MORSE, A. S., 1970, Decoupling and pole assignment in linear multivariable systems: a geometric approach. *SIAM Journal of Control*, **8**, 1–18.