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A THEORY OF STABILIZATION FOR NONLINEAR SYSTEMS *

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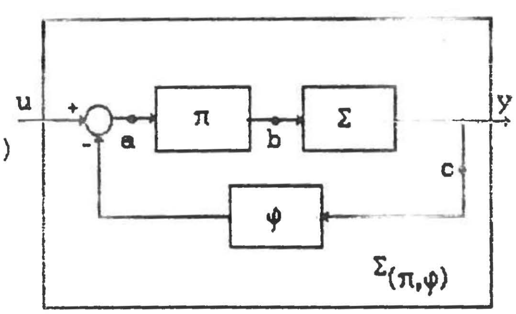
Abstract

We show that a theory of control and stabilization can be developed for nonlinear systems, using fraction representations and coprimeness as the basic mathematical notions. The theory bears striking resemblance to the transfer matrix theory of linear systems.

1. INTRODUCTION AND BASICS

The objective of our present paper is to review the fundamental aspects of a theory of stabilization of nonlinear systems recently developed by the author in HAMMER [1986a and b]. We shall provide here only a qualitative and brief exposition of the main results, and we shall omit all proofs. Of course, the complete discussion of our results and the proofs can be found in the references.

Let Σ be a strictly causal nonlinear system which has to be stabilized. To stabilize Σ, we connect it in the classical control configuration

(1.1)


where π is a causal dynamic precompensator, and where φ is a causal dynamic feedback compensator. We denote by $\Sigma_{(\pi,\varphi)}$ the overall system described by the diagram. We wish to study the following

(1.2) PROBLEM. For a given system Σ, find all pairs of causal compensators π and φ for which $\Sigma_{(\pi,\varphi)}$ is internally stable.

The basic tool that we use in our investigation of (1.2) is the theory of fraction representations of nonlinear systems developed in HAMMER [1986b and 1985a], the basic aspects of which we review below. First, however, we describe in more precise terms the nature of the systems we wish to consider, and the meaning of stability.

Our framework is presently built for the study of discrete-time systems. As usual, we denote by R the set of real numbers, and by R^m, where $m > 0$ is an integer, the set of all m-tuples of real numbers. By $S_0(R^m)$ we denote the set of all infinite sequences $u = \{u_0, u_1, u_2, \dots\}$, where $u_j \in R^m$ for all integers $j \geq 0$, and where we interpret the index j as the time marker. Thus, $S_0(R^m)$ is simply the set of all time-sequences of m-dimensional real vectors, starting at time zero. Given a sequence $u \in S_0(R^m)$ and an integer $i \geq 0$, we denote by u_i the i-th element of the sequence. In the space $S_0(R^m)$ we define the (standard) operation of addition elementwise, so that for every pair of sequences $u, v \in S_0(R^m)$, the sum sequence $w := u + v$ has the elements $w_i = u_i + v_i$ for all integers $i \geq 0$.

A system Σ is simply a map $\Sigma : S_0(R^m) \to S_0(R^p)$, transforming input sequences of m-dimensional vectors into output sequences of p-dimensional vectors. By composition of systems we simply mean the composition of the maps representing the systems. The addition of systems is defined pointwise in the usual way, so that, given two systems $\Sigma_1, \Sigma_2 : S_0(R^m) \to S_0(R^p)$, their sum $\Sigma := \Sigma_1 + \Sigma_2 : S_0(R^m) \to S_0(R^p)$ transforms every input sequence $u \in S_0(R^m)$ into the output sequence $\Sigma u := \Sigma_1 u + \Sigma_2 u \in S_0(R^p)$.

From the practical point of view, the most interesting input sequences are, of course, the bounded ones. To treat bounded input sequences we denote, for every real $\theta > 0$, by $[-\theta,\theta]^m$ the set of all real vectors in R^m with entries in the interval $[-\theta,\theta]$. We denote by $S_0(\theta^m)$ the set of all sequences $u \in S_0(R^m)$ for which $u_i \in [-\theta,\theta]^m$ for all integers $i \geq 0$. Thus, $S_0(\theta^m)$ is the set of all input sequences 'bounded' by θ.

We now turn to stability, and we first review several concepts related to boundedness and to continuity. Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a system. As usual, we denote by $\mathrm{Im}\,\Sigma$ the subset of $S_0(R^p)$ consisting of all possible output sequences of Σ. Given a set $S \subset S_0(R^m)$, we denote by $\Sigma[S]$ the set of all possible output sequences of the restriction of Σ to S. We say that the system Σ is BIBO (Bounded-Input Bounded-Output) -stable if, for every real $\theta > 0$, there exists a real $M > 0$ such that $\Sigma[S_0(\theta^m)] \subset S_0(M^p)$. In other words, a BIBO-stable system transforms bounded input sequences into bounded output sequences. Next, in order to discuss continuity, we define a metric on our spaces of sequences. For a vector $a = (a^1, ..., a^m) \in R^m$, we denote $|a| := \max \{ |a^1|, ..., |a^m| \}$. On the set $S_0(R^m)$, we define a norm ρ given, for every element $u \in S_0(R^m)$, by $\rho(u) := \sup_{i \geq 0} 2^{-i}|u_i|$. The norm ρ induces a metric ρ on $S_0(R^m)$ when, for every pair of elements $u, v \in S_0(R^m)$, one sets $\rho(u,v) := \rho(u-v)$. Unless explicitly stated otherwise, all references to continuity below refer to continuity with respect to the topology induced by the metric ρ. A system $\Sigma : S_0(R^m) \to S_0(R^p)$ is stable if it is BIBO-stable and if, for every real $\theta > 0$, its restriction $\Sigma : S_0(\theta^m) \to S_0(R^p)$ is a continuous map. This definition of stability is, of course, in the spirit of the Liapunov notion of stability.

Much of our discussion in this work deals with composite systems. When talking about stability of composite systems, we have to distinguish between two concepts of stability - input/output stability and internal stability. We say that $\Sigma_{(\pi,\varphi)}$ is input/output stable if the input/output relation represented by $\Sigma_{(\pi,\varphi)}$ is stable: We say that $\Sigma_{(\pi,\varphi)}$ is intenally stable if it is input/output stable, if all its internal signals are bounded, and if small additive noises activated at the points a, b, and c in diagram (1.1) cause only small changes in the output y of the composite system (see HAMMER [1986a] for an accurate definition). All composite systems that we consider in our discussion are internally stable.

The basic underlying concept of our discussion is the concept of fraction representations of nonlinear systems. Given a nonlinear system $\Sigma : S_0(R^m) \to S_0(R^p)$, we say that Σ has a right fraction representation if there is an integer $q > 0$, a subspace $S \subset S_0(R^q)$, and a pair of stable maps $P : S \to \mathrm{Im}\,\Sigma$ and $Q : S \to S_0(R^m)$, where Q is invertible, such that $\Sigma = PQ^{-1}$. The subspace S is called the factorization space of the fraction representation $\Sigma = PQ^{-1}$. We say that Σ has a left fraction representation if there is an integer $q > 0$, a subspace $S' \subset S_0(R^q)$, and a pair of stable maps $G : \mathrm{Im}\,\Sigma \to S'$ and $T : S_0(R^m) \to S'$, where G is invertible, such that $\Sigma = G^{-1}T$. This definition of fraction representations is reminiscent of one of the most common procedures used in linear system theory - the expression of a transfer function F of a linear time-invariant system as a quotient of two polynomial matrices. One of the main themes of our discussion is that fraction representations are of pivotal importance not only to the well known theory of linear systems, but to the theory of nonlinear systems as well. As we shall show later, most nonlinear systems of common practical interest do possess right and left fraction representations.

Returning now to (1.1), assume that $\Sigma : S_0(R^m) \to S_0(R^p)$ is a strictly causal system, and that $\pi : S_0(R^m) \to S_0(R^m)$ and $\varphi : S_0(R^p) \to S_0(R^m)$ are causal systems. Then, the overall system described by the diagram can be expressed by the formula (e.g., HAMMER [1984b])

$$(1.3) \quad \Sigma_{(\pi,\varphi)} = \Sigma\pi[I + \varphi\Sigma\pi]^{-1},$$

where $I : S_0(R^m) \to S_0(R^m)$ is the identity system. From our present point of view, the most interesting case occurs when the precompensator π and the feedback compensator φ are chosen in the particular form

$$(1.4) \quad \begin{aligned} \varphi &= A, \\ \pi &= B^{-1}, \end{aligned}$$

where $A : S_0(R^p) \to S_0(R^m)$ is a stable and causal system, and where $B : S_0(R^m) \to S_0(R^m)$ is a stable and invertible system having a causal inverse. Assume now that Σ has a right fraction representation $\Sigma = PQ^{-1}$, where $P : S \to \operatorname{Im} \Sigma$ and $Q : S \to S_0(R^m)$, and where $S \subset S_0(R^q)$. Then, substituting into (1.3), we obtain

$$(1.5) \quad \Sigma_{(\pi,\varphi)} = PQ^{-1}B^{-1}[I + APQ^{-1}B^{-1}]^{-1} = P[BQ + AP]^{-1},$$

an equation that reminds us of the situation in the linear case. Clearly, if we can find stable systems A and B (subject to the aforementioned requirements) for which the stable and invertible system

$$(1.6) \quad M := AP + BQ \;:\; S \to S$$

has a stable inverse M^{-1}, then, upon setting $\varphi = A$ and $\pi = B^{-1}$, the overall system $\Sigma_{(\pi,\varphi)} = PM^{-1}$ becomes input/output stable. Moreover, as we discuss later, a slight strengthening of the stability requirements imposed on the systems A and B will actually guaranty that the system $\Sigma_{(\pi,\varphi)}$ is not just input/output stable, but internally stable as well. We arrive then at the following fundamental

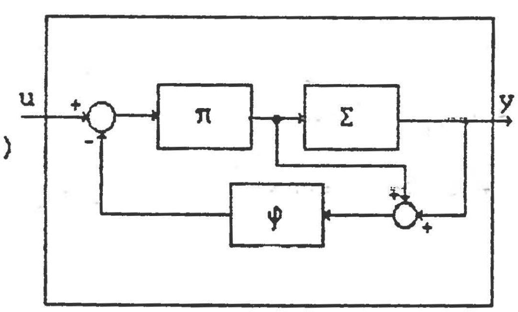
(1.7) QUESTION. Given a pair of stable maps $P : S \to S_0(R^p)$ and $Q : S \to S_0(R^m)$, where $S \subset S_0(R^q)$, when does there exist a pair of stable maps $A : S_0(R^p) \to S_0(R^q)$ and $B : S_0(R^m) \to S_0(R^q)$ such that the stable map $M := AP + BQ : S \to S$ has a stable inverse M^{-1}. When such a pair exists, find all pairs of stable maps $A' : S_0(R^p) \to S_0(R^q)$ and $B' : S_0(R^m) \to S_0(R^q)$ satisfying $M = A'P + B'Q$.

For the case of linear systems, Question (1.7) has a well known answer in the theory of matrices over Euclidian domains. As it turns out, a closely analogous theory can be constructed for the case of nonlinear systems as well, as we discuss below.

2. COPRIMENESS AND FRACTION REPRESENTATIONS

We wish to discuss first two basic assumptions that we make regarding the system Σ which is to be stabilized. Formally, our first assumption is that the system Σ which has to be stabilized is operated only by bounded input sequences, namely, that there is a fixed, but otherwise arbitrary, number $\alpha > 0$ such that $\Sigma : S_0(\alpha^m) \to S_0(R^p)$. This assumption needs little explanation, since most common practical systems have an inherent bound on the maximal allowable amplitude of the input signal, above which the mathematical model of the system is violated.

Our second assumption is that the system Σ which needs to be stabilized is an injective system. This assumption requires explanation, since most common systems are, of course, not injective. However, we show in HAMMER [1986b] that, through a minor change in the control configuration (1.1), the problem of stabilizing any strictly causal system Σ can be transformed into the problem of stabilizing an injective system. Basically, this is done by stabilizing the system $\Sigma_e \approx I+\Sigma$, the sum of Σ and the identity system, instead of stabilizing the system Σ directly. Using causality considerations, it is easy to see intuitively that the system $I+\Sigma$ is injective whenever the given system Σ is strictly causal. Though the expression $I+\Sigma$ may formally be invalid due to possible discrepancies in the dimensions of the input and output spaces of Σ, this difficulty can be readily settled (HAMMER [1986b]). Now, when one uses the configuration (1.1) to stabilize the system Σ_e, one also obtains stabilization of the system Σ in a slightly different control configuration, which qualitatively looks as the one in diagram (2.1) below.

(2.1)


Thus, the restriction to injective systems simply amounts to replacing the configuration (1.1) by the configuration (2.1) when Σ is not injective. To summarize, we have seen that from the control theoretic point of view, we can limit ourselves to studying the stabilization of <u>injective</u> systems $\Sigma : S_0(\alpha^m) \to S_0(R^p)$, where $\alpha > 0$ is a fixed, but otherwise arbitrary, real number.

Returning now to Question 1.7, we reproduce from HAMMER [1985a and 1986b], the definition of right coprimeness. Qualitatively, two stable maps P and Q are right coprime if, for every unbounded input sequence u, at least one of the output sequences Pu or Qu is unbounded. In the linear case, this would reduce to the requirement that P and Q have no unstable zeros in common. (Given a map $P : S_o(R^m) \to S_o(R^p)$ and a subset $S \subset S_o(R^p)$, we denote by $P^*[S]$ the inverse image of the set S through P, i.e., the set of all elements $u \in S_o(R^m)$ satisfying $Pu \in S$.)

(2.2) DEFINITION. Let $S \subset S_o(R^q)$ be a subset. Two stable maps $P : S \to S_o(R^p)$ and $Q : S \to S_o(R^m)$ are right coprime if the following conditions hold:
(i) For every real $\tau > 0$ there exists a real $\theta > 0$ such that $P^*[S_o(\tau^p)] \cap Q^*[S_o(\tau^m)] \subset S_o(\theta^q)$, and
(ii) For every real $\tau > 0$, the set $S \cap S_o(\tau^q)$ is a closed subset of $S_o(\tau^q)$.[]

As we recall, our interest in (1.7) was motivated by (1.5), so that the maps P and Q in our discussion originate from a fraction representation $\Sigma = PQ^{-1}$ of the system Σ that has to be stabilized. In the next result, taken from HAMMER [1986b], we show that when P and Q are right coprime, the maps A and B of (1.7) can be found.

(2.3) THEOREM. Let $\Sigma : S_o(\alpha^m) \to S_o(R^p)$ be an injective system, and assume it has a right fraction representation $\Sigma = PQ^{-1}$, where $P : S \to \mathrm{Im}\,\Sigma$ and $Q : S \to S_o(\alpha^m)$, and where $S \subset S_o(R^q)$. If P and Q are right coprime, then, for every stable map $M : S \to S$, there is a pair of stable maps $A : \mathrm{Im}\,\Sigma \to S_o(R^q)$ and $B : S_o(\alpha^m) \to S_o(R^q)$ satisfying $AP + BQ = M$.

Theorem 2.3 opens the following new question.

(2.4) QUESTION. When does an injective system $\Sigma : S_o(\alpha^m) \to S_o(R^p)$ possess a representation $\Sigma = PQ^{-1}$, where P and Q are stable and right coprime maps.

As it turns out, not every injective system $\Sigma : S_o(\alpha^m) \to S_o(R^p)$ possesses a right coprime fraction representation. The only systems possessing such a representation are the so called 'homogeneous' systems (HAMMER [1985a and 1986b]), which, qualitatively speaking, satisfy the condition of being continuous over sets of inputs which produce bounded outputs. The exact definition is as follows.

(2.5) DEFINITION. A system $\Sigma : S_o(R^m) \to S_o(R^p)$ is a homogeneous system if for every real $\alpha > 0$ the following holds: for every subset $S \subset S_o(\alpha^m)$ for which there exists a real $\theta > 0$ such that $\Sigma[S] \subset S_o(\theta^p)$, the restriction of Σ to the closure $\overline{S}$ of S in $S_o(\alpha^m)$ is a continuous map $\Sigma : \overline{S} \to S_o(\theta^p)$.[]

The class of homogeneous systems is identical to the class of systems possessing right coprime fraction representations (HAMMER [1986b]).

(2.6) THEOREM. An injective system $\Sigma : S_o(\alpha^m) \to S_o(R^p)$ has a right coprime fraction representation if and only if it is a homogeneous system.

In HAMMER [1985a] we showed that Theorem 2.6 also holds for injective systems $\Sigma : S_o(R^m) \to S_o(R^p)$. Fortunately, many systems of common engineering interest are homogeneous systems. We now provide an example of a rather large class of such systems. A system $\Sigma : S_o(R^m) \to S_o(R^p)$ is recursive if there exists a pair of integers $\eta, \mu \geq 0$ and a function $f : (R^p)^{\eta+1} \times (R^m)^{\mu+1} \to R^p$ such that, for every input sequence $u \in S_o(R^m)$, the output sequence $y := \Sigma u$ satisfies $y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu})$ for all integers $k \geq 0$. The initial conditions $y_0, \ldots, y_\eta$ must be specified and fixed. The function f is called a recursion function of Σ. It can be shown (HAMMER [1985b, 1986b]) that every recursive system having a continuous recursion function is a homogeneous system. Thus, our theory applies to most systems encountered in common engineering practice.

The main objective of the theory of right coprimeness is to provide us with one pair of stable systems A, B satisfying $AP + BQ = M$, whenever P and Q are right coprime. The question of finding all pairs of stable systems A, B satisfying the equation $AP + BQ = M$ requires some further consideration. Crucial to the solution of this latter question is the theory of left fraction representations of nonlinear systems developed in HAMMER [1986b]. It is rather easy to see how left fraction representations enter into the discussion, as follows.

Let $\Sigma : S_o(\alpha^m) \to S_o(R^p)$ be a homogeneous system, let $\Sigma = PQ^{-1}$ be a right coprime fraction representation of Σ, and assume that Σ also has a left fraction representation $\Sigma = G^{-1}T$. Then, we clearly have $G^{-1}T = PQ^{-1}$, or $TQ = GP$. Assume further that one pair of stable systems A, B satisfying $AP + BQ = M$ is known. To find additional pairs of stable systems A, B satisfying the same equation, we can proceed in a manner closely resembling linear methods. We choose an arbitrary stable system h, having appropriate input and output spaces, and we define the pair of stable systems

$$(2.7)\quad \begin{aligned} A' &= A - hG, \\ B' &= B + hT. \end{aligned}$$

Then, since $hTQ = hGP$, we have $A'P + B'Q = (A - hG)P + (B + hT)Q = AP + BQ + (hTQ - hGP) = AP + BQ = M$, and we obtained a new pair of stable systems A', B' satisfying $A'P + B'Q = M$. In fact, infinitely many pairs of stable systems A', B' satisfying $A'P + B'Q = M$ can be obtained in this way, one pair for each choice of h. Moreover, using this simple method, one can actually obtain all solutions of the equation $A'P + B'Q = M$. Thus, we may conclude that left fraction representations are of fundamental significance to the stabilization problem of nonlinear systems.

The theory of left fraction representations of homogeneous injective systems $\Sigma : S_o(\alpha^m) \to S_o(R^p)$ is rather simple, due to the compactness of the domain $S_o(\alpha^m)$. We start our review of this theory with the following result (HAMMER [1986b]).

(2.8) THEOREM. _An injective homogeneous system_ $\Sigma : S_o(\alpha^m) \to S_o(R^p)$ _has a left fraction representation._

One of the most surprising facts in the theory of left fraction representations of injective homogeneous systems $\Sigma : S_o(\alpha^m) \to S_o(R^p)$ is that every left fraction representation $\Sigma = G^{-1}T$ of such a system is actually a 'coprime' left fraction representation in an intuitive sense, and the numerator system T has a stable inverse. This fact is a major departure from the theory of linear systems, and it originates from the compactness of the domain $S_o(\alpha^m)$. (The domain $S_o(\alpha^m)$ cannot be used in the linear case since it violates the linearity of the input space.) The following, combined with Theorem 2.3, provides a complete answer to Question 1.7 (HAMMER [1986b]).

(2.9) THEOREM. _Let_ $\Sigma : S_o(\alpha^m) \to S_o(R^p)$ _be an injective homogeneous system, and let_ $\Sigma = PQ^{-1}$ _be a right coprime fraction representation, where_ $P : S \to \mathrm{Im}\,\Sigma$ _and_ $Q : S \to S_o(\alpha^m)$, and where $S \subset S_o(R^q)$. _Let_ $\Sigma = G^{-1}T$ _be a left fraction representation of_ Σ, _where_ $G : \mathrm{Im}\,\Sigma \to S_L$ _and_ $T : S_o(\alpha^m) \to S_L$. _Let_ $M : S \to S$ _be any stable map, and let_ $A : \mathrm{Im}\,\Sigma \to S_o(R^q)$ _and_ $B : S_o(\alpha^m) \to S_o(R^q)$ _be a pair of stable maps satisfying the equation_ $AP + BQ = M$. _Then, a pair of stable maps_ $A' : \mathrm{Im}\,\Sigma \to S_o(R^q)$ _and_ $B' : S_o(\alpha^m) \to S_o(R^q)$ _satisfies_ $A'P + B'Q = M$ _if and only if there exists a stable map_ $h : S_L \to S_o(R^q)$ _such that_ $A' = A - hG$, _and_ $B' = B + hT$.

Finally, we remark that explicit constructions for fraction representations of nonlinear systems and for systems A and B satisfying $AP + BQ = M$ are given in HAMMER [1984a, 1985a, 1986a, and 1986b].

3. INTERNAL STABILIZATION

We consider the internal stabilization of an injective and causal nonlinear system $\Sigma : S_o(R^m) \to S_o(R^p)$ using the control configuration (1.1), with the particular form of the compensators π and φ of (1.4), so that $\Sigma(\pi,\varphi) = \Sigma(B^{-1},A)$. As before, we assume that the stabilized system $\Sigma(\pi,\varphi)$ is operated only by bounded inputs, namely, that there is a real number $\theta > 0$ such that all input sequences u of $\Sigma(\pi,\varphi)$ are taken from $S_o(\theta^m)$. Before stating our results on internal stabilization, we have to introduce some terminology. Let $S_1 \subset S_o(R^m)$ and $S_2 \subset S_o(R^p)$ be two subsets, and let $M : S_1 \to S_2$ be a map. We say that M is _unimodular_ if M has an inverse M^{-1} and if M and M^{-1} are both stable maps; if M is unimodular, we say that the sets S_1, S_2 are _S_ (stability) -_morphic_. Also, for a pair of elements $u, v \in S_o(R^m)$, we denote $|u - v| := \sup_{i \geq o}\{|u_i - v_i|\}$. Next, we need the following (HAMMER [1986a])

(3.1) DEFINITION. Let $A : S_o(R^m) \to S_o(R^p)$ be a stable map, and let $\theta > 0$ be a real number. We say that A is _differentially bounded by_ θ if there exists a real $\varepsilon > 0$ such that, for every pair of elements $y, y' \in S_o(R^m)$ satisfying $|y - y'| < \varepsilon$, one has $|A(y) - A(y')| < \theta$.[]

We note that, for a differentially bounded map, a bounded (by ε) fluctuation of the input sequence causes only a bounded (by θ) fluctuation of the output sequence. As an example of a class of differentially bounded maps, we have the class of maps which are uniformly ℓ^∞-continuous (HAMMER [1986a]). The next result, also taken from HAMMER [1986a], shows that the problem of internal stabilization of Σ, using the configuration (1.1) with the compensators π and φ of (1.4), actually reduces to the problem of finding a pair of stable and differentially bounded systems A and B satisfying the equation $AP + BQ = M$, where M is a unimodular transformation. Thus, the problem of internal stabilization is reduced to studying the solutions A, B of the equation $AP + BQ = M$, in close analogy to the situation in the linear case. The Theorem makes the assumption that the factorization space S of a right coprime fraction representation of Σ, contains a subspace S' which is S-morphic to $S_0((5\theta)^m)$. This assumption, which is a necessary condition for internal stabilization of the system Σ, was studied in detail in HAMMER [1986a]). As we mention later, most practical systems satisfy this assumption.

(3.2) THEOREM. Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a causal homogeneous system, and let $\theta > 0$ be a real number. Let $\Sigma = PQ^{-1}$ be a right coprime fraction representation, and let $S \subset S_0(R^q)$ be its factorization space. Assume S contains a subset S' which is S-morphic to $S_0((5\theta)^m)$, and let $M : S' \to S_0((5\theta)^m)$ be a unimodular transformation. Assume there is a pair of stable maps $A : S_0(R^p) \to S_0(R^m)$ and $B : S_0(R^m) \to S_0(R^m)$ satisfying the equation $APv + BQv = Mv$ for all elements $v \in S'$, where A and B are differentially bounded by θ, A is causal, and B is bicausal. Then, the composite system $\Sigma(B^{-1},A)$ is internally stable for all input sequences $u \in S_0(\theta^m)$.

Before continuing with our discussion, we need some terminology. The first term is related to causality properties. Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective system, and denote by $\mathcal{D}$ the one-step time-delay operator. Also, let $\Sigma^{-1} : \mathrm{Im}\,\Sigma \to S_0(R^m)$ be the inverse system of Σ (which exists by the injectivity of Σ). We say that Σ is a normal system if there is an integer n such that $\mathcal{D}^n\Sigma^{-1}$ is a causal system. Most systems in nature are normal systems. For instance, recursive systems are normal. Next, we have the following

(3.3) DEFINITION. Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective homogeneous system. A subset $S \subset S_0(R^m)$ is a stability subspace of Σ if there is a pair of real numbers $\alpha, \beta > 0$ such that $S \subset S_0(\alpha^m)$ and $\Sigma[S] \subset S_0(\beta^p)$. A stability subspace $S \subset S_0(\alpha^m)$ of Σ is full if there is a bicausal, stable, and uniformly ℓ^∞-continuous map $M : S_0(R^m) \to S_0(R^m)$ and a real number $\gamma > 0$ such that $M[\overline{S}] = S_0(\gamma^m)$, where $\overline{S}$ is the closure of S in $S_0(\alpha^m)$.[]

Again, most common practical systems do possess a full stability subspace, as we impart briefly below. A detailed discussion of full stability subspaces is provided in HAMMER [1986a]. We can now list a main result on intenal stabilization of nonlinear systems.

(3.4) THEOREM. Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a homogeneous, causal, normal, and injective system, having a full stability subspace. Then, for any real number $\theta > 0$, there is a pair of stable systems $A : S_0(R^p) \to S_0(R^m)$ and $B : S_0(R^m) \to S_0(R^m)$, where A is causal and B is bicausal, such that the system $\Sigma(B^{-1},A)$ is internally stable for all input sequences $u \in S_0(\theta^m)$.

We emphasize that our discussion in HAMMER [1986a] also contains constructions for the systems A and B of Theorem 3.4. We also show there that, generically, every causal recursive (injective) system $\Sigma : S_0(R^m) \to S_0(R^p)$ having a twice continuously differentiable recursion function, satisfies the conditions of Theorem 3.4. Other systems may also satisfy these conditions. Thus, we may say that our theory applies to most systems of practical interest.

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